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Research Article

On the homology of Borel subgroup of SL(2,Fp)

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ABSTRACT
In the theory of algebraic groups, a Borel subgroup of an algebraic group is a maximal Zariski closed and connected solvable algebraic subgroup. In the case of the special linear group SL_2 over finite fields $\mathbb{F}_p$ the subgroup of invertible upper triangular matrices B is a Borel subgroup. According to Adem [1], these are periodic groups. In this paper we compute the integral homology of the Borel subgroup B of the special linear group $\mathrm{SL}(2,\mathbb{F}_p)$ where p is a prime. In order to compute the integral homology of B, we decompose it into $\ell-$ primary parts. We compute the first summand based on Invariant Theory and compute the rest based on Lyndon-Hochschild-Serre spectral sequence. In conclusion, we found the presentation of B and its period. Furthermore, we also explicitly compute the integral homology of B.
Key words: Ring cohomology of p-groups, periodic groups, Invariant Theory, Lyndon-Hochschild-Serre spectral sequence

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PRELIMINARIES

For reference, we briefly recite some facts about group cohomology and the transfer homomorphism [1–4], which will be used frequently throughout this paper.
Let G be a finite group and A be a $G-$module, then we define

$$H^n(G,A) := H^n(B_G,A)$$

where B_G is classifying space of the group G. The group $H^n(G,A)$ is called the *cohomology group of G with (untwisted) coefficient A*. If $H \subset G$ is a subgroup, the inclusion $B_H \to B_G$ induces a map in cohomology

$$res^G_H : H^n(G,A) \to H^n(H,A)$$

called *restriction*. Because inner automorphisms of G act trivially on cohomology, we have $Im(res^H_G)$ is contained in $H^n(H,A)^{N_G(H)/H}$. There is also a *transfer map* going other way,

$$tr^G_H : H^n(H,A) \to H^n(G,A).$$

They are related by two composition formulae.

- $tr^H_G \circ \mathrm{res}^G_H$ equals multiplication by $[G:H]$ on $H^n(G,A)$.
- (Double coset formula)

$$\mathrm{res}^G_H \circ tr^G_K = \sum_i tr_{H\cap x_iKx_i^{-1}} \circ \sum_i \mathrm{res}^{x_iKx_i^{-1}}_{H\cap x_iK_i} \circ c_{x_i}$$

where $K \subset G$ is also a subgroup, the sum is over double-coset representatives, and $C_x : xHx^{-1} \to H$ is conjugation.
Some consequences of the two formulae.

- If p does not divide G, then $H^n\left(G,\mathbb{F}_p\right) = 0$ for all $n > 0$
- If H is contains a Sylow $p-$ subgroup of G, then

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$$\mathrm{res}_H^G : H^n(G,A)_{(p)} \to H^n(H,A)_{(p)}$$

is injective, where the subscipt is $p-$primary part.

- If H contains a Sylow $p-$ subgroup of G and is normal in G, then

$$res_H^G : H^n(G,A)_{(p)} \cong H^n(H,A)^{G/H}.$$

- If G is an elementary abelian $p-$ group and H is a proper subgroup, then

$$tr_H^G : H^n(H,A) \to H^n(G,A)$$

is zero.
Let G be a finite group and A be a $G-$module, then we define

$$H_n(G,A) := H_n(B_G,A)$$

where B_G is classifying space of the group G. The group $H_n(G,A)$ is called the *homology group of G with (untwisted) coefficient A*. Take $n = 1$ and $A = \mathbb{Z}$, there is a canonical isomorphism

$$H_1(G,\mathbb{Z}) \cong G/[G,G] \tag{1}$$

where $[G,G]$ is commutator subgroup of G.
When $H \subset G$ is a normal subgroup, there is a Lyndon-Hochschild-Serre spectral sequence

$$H_p\left(G/H, H_q(H,A)\right) \Rightarrow H_{p+q}(G,A).$$

The following two facts are the best tool to change of ring or to change between cohomology and homology.

- (Universal coefficient theorem for group homology)

$$H_p(G,A) \cong \left(H_p(G,\mathbb{Z}),A\right) \oplus \mathrm{Tor}\left(H_{p-1}(G,\mathbb{Z}),A\right).$$

- (Dual coefficient theorem for group cohomology)

$$H^p(G,A) \cong \mathrm{Hom}\left(H_p(G,\mathbb{Z}),A\right) \oplus \mathrm{Ext}\left(H_{p-1}(G,\mathbb{Z}),A\right).$$

THE PRESENTATION AND THE PERIODICITY OF BOREL SUBGROUP OF $SL(2,\mathbb{F}_P)$

Let $G = SL\left(2,\mathbb{F}_p\right)$. Let B be the subgroup of upper triangular matrices in G, D the subgroup of diagonal matrices in G, and U the subgroup of upper triangular matrices with all their diagonal coefficients equal to 1. We describe these group as follow

$$B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}, D = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}, U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}.$$

Then, U is normal subgroup of $B, UD = B$, and $U \cap D = \{1\}$[5–8]. The group B is called the *Borel subgroup* of G and $B = UD$, a semidirect product. To find the presentation, we need the following lemmas.

Lemma 1 (5.4.5) [9] *Let G be a group of finite order N in which every Sylow subgroups is cyclic. Then G is generated by two elements A and C with defining relations*

$$\begin{array}{l} A^m = C^n = I, \quad CAC^{-1} = A^{'},\, N = nm \\ ((r-1)n, m) = 1,\ r^n = 1\ (\mod m) \end{array}.$$

Lemma 2 *Let B be the Borel subgroup of* $SL(2,\mathbb{F}p)$. *Then every Sylow subgroups of B is cyclic.*

Proof

Firstly, the subgroup U is the Sylow $p-$subgroup B and this group is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In case $\ell | p-1$. The Sylow $l-$subgroups is the subgroups of D. Since D is cyclic, these groups are also cyclic.

Proposition 3 *Let B be a Borel subgroup of* $SL(2,\mathbb{F}p)$. *Then*

$$B = \left\langle T, y_a | T^p = I, y_a^{p-1} = I, y_a T y_a^{-1} = T^{a^2} \right\rangle,$$

where a is a generator of $(\mathbb{Z}/p)^*$. Moreover,

$$H_1(B) \cong \mathbb{Z}/(p-1).$$

Proof. We begin with the first observation

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & xa^{-1} \\ 0 & a^{-1} \end{pmatrix}$$

where $x \in \mathbb{Z}/p$ and $a \in (\mathbb{Z}/p)^*$.

Since $(\mathbb{Z}/p)^*$ is a group, an element x^{a-1} runs through all a set $\{0,1,\ldots,p-1\}$ when x runs through $\mathbb{Z}/p$. By set $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = T^x.$$

Next, by $B \cong \mathbb{Z}/p \times \mathbb{Z}/(p-1)$, we get

$$y_a x y_a^{-1} = x^{a^2},$$

where $y_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Now let a be a generator element of $(\mathbb{Z}/p)^*$. Since y_a is the diagonal matrix, we get

$$\langle y_a \rangle = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} | b \in (\mathbb{Z}/p)^* \right\}.$$

Therefore,

$$B = \left\langle T, y_a | T^p = I, y_a^{p-1} = I, y_a T y_a^{-1} = T^{a^2} \right\rangle$$

these relations are maximum by Lemma 1 (Notice that we can apply Lemma 1 since B satisfies Lemma 2). Hence, by (1) we get

$$H_1(B) = B/\langle B,B \rangle = \left\langle \overline{T}, \overline{y_a} | \overline{T}^p = I, \overline{y_a}^{p-1} = I, \overline{T}^{a^2-1} = I \right\rangle.$$

Assume $a^2 - 1 \equiv k(\mod p)$. Then $(k,p) = 1$, now there are r,t such that

$$kr + pt = 1,$$

hence,

$$\overline{T}^{kr+pt} = \overline{T}.$$

This implies that $T = 1$. Thus we get $H_1(B) \cong \mathbb{Z}/(p-1)$.
The finite group G is *periodic* and of period $n > 0$ if and only if $H^i(G,\mathbb{Z}) \cong H^{i+n}(G,\mathbb{Z})$ for $i \geq 1$. Arcoding to Thomas[10] the group G is a periodic group if and only if a $p-$Sylow subgroup is either cyclic or generalised quaternion/binary dihedral (if $p = 2$). From Lemma 2, we get B is the periodic groups but the period is unknown.
The $p-period$ of group G is the period of $H^n(G,\mathbb{Z})_{(p)}$. When p is odd, it is easy to calculate the $p-$period.
Lemma 4 ([10]) *Let N_p be the normalizer of the $p-$Sylow subgroup of G and Z_p its centralizer. Then the $p-$period is $2\left[N_p : Z_p\right]$.*
Proposition 5 *The $p-$period of B is $p-1$.*
Proof. The Sylow $p-$ subgroup of B is the subgroup U. By the Proposition 3, its normalizer is the whole group B and its centralizer is generated by $-I, U$. By Lemma 5, we get the $p-$period is

$$2(p(p-1)/2p) = p-1,$$

since $|Np| = p(p-1)$ and $|Zp| = 2p$.

THE INTEGRAL HOMOLOGY OF BOREL SUBGROUPS B OF $SL(2,\mathbb{Z}/P)$

In this section we will give a detailed computation of the Borel subgroup of $SL(2,Fp)$. In order to compute the integral homology of B, we decompose it into ℓ-primary parts

$$H_n(B,\mathbb{Z}) = \oplus_{\ell | order(B)} H_n(B,\mathbb{Z})_{(\ell)} = H_n(B,\mathbb{Z})_{(p)} \oplus \left(\oplus_{q\neq p}\right) H_n(B,\mathbb{Z})_{(q)}.$$

To compute the first summand, we are concerned with the ring cohomology of $p-$groups. In cohomology there is a cup product induced from the diagonal map and the Kunneth formula. In particular, with (untwisted) field coefficients $\mathbb{F}$, this gives a pairing

$$\sum_{i,j} H^i(G,\mathbb{F}) \otimes_{\mathbb{F}} H^j(G,\mathbb{F}) \to H^{i+j}(G,\mathbb{F})$$

making $H^*(G,\mathbb{F}) := \sum_i H^i(G,\mathbb{F})$ into an associated commutative ring with unit. The following lemma gives us the structure of ring cohomology of $p-$groups.
Lemma 6 ([1]) *Let p be an odd prime, then $H^*\left(\mathbb{Z}/p,\mathbb{F}_p\right) = E\left(v_1\right) \otimes \mathbb{F}_p\left[b_2\right]$, the tensor product of a polynomial algebra on a two dimensional generator and an extorior algebra on a $1-$dimensional generator.*
Theorem 7

$$H_k(B,\mathbb{Z})_{(p)} = \begin{cases} 0 \text{ otherwise} \\ \mathbb{Z}/p \quad \text{if} \quad k \equiv 0(p-2) \end{cases}.$$

Proof. Let B be a Borel subgroup of $SL(2,\mathbb{F}p)$. Then the $p-$Sylow subgroup is $U \cong \mathbb{Z}/p$, this group is also normal in B, so we have

$$H^*(B,\mathrm{F_p}) \ = \ \mathrm{H^*(\mathbb{Z}/p,F_p)^{F_p^*}}.$$

Using ring structure from Lemma 6

$$H^*\left(\mathbb{Z}/p,\mathbb{F}_p\right)^{\mathbb{Z}/p} = \left(E(x) \otimes \mathbb{F}_p[y]\right)^{\mathbb{F}_p^*}$$

with y in cohomological degree 2 and x in cohomological degree 1. The action is multiplicative and determined by $ax := a^2x, ay := a^2y$ $\left(a \text{ is generator of } \mathbb{Z}_p^*\right)$.
The elements of $E(x) \otimes \mathbb{F}_p[y]$ only have the forms $\sum_{i=0}^{i=p-1} a_i y^i$ and $\sum_{i=0}^{i=p-1} a_i x y^i$ cause $x^2 = 0 \left(a_i \in \mathbb{F}_p\right)$. Under the above action, we have

$$a\left(\sum_{i=0}^{i=p-1} a_i y^i\right) = \sum_{i=0}^{i=p-1} a_i \left(a^2 y\right)^i, \text{ and } \quad a\left(\sum_{i=0}^{i=p-1} a_i x y^i\right) = \sum_{i=0}^{i=p-1} a_i \left(a^2 x\right)\left(a^2 y\right)^i.$$

By Fermats little Theorem, the invariant must be generated by $y^{(p-1)/2}$ and $y^{(p-3)/2}x$. It implies that $\mathbb{Z}/p$ just appear in the position $k = 0\; mod(p-2)$ or $k = 0\; mod(p-1)$. Therefore, for $k \neq 0$

$$H^k(B,\mathbb{Z})_{(p)} = \begin{cases} 0\; otherwise \\ \mathbb{Z}/p \\ if\; k \equiv 0(p-2) \\ \mathbb{Z}/p \\ if\; k \equiv 0(p-1) \end{cases}.$$

Now using the Dual coefficient theorem for group cohomology, we obtain

$$H_k(B,\mathbb{Z})_{(p)} = \begin{cases} 0 \text{ otherwise} \\ \mathbb{Z}/p \text{ if } k \equiv 0(p-2) \end{cases}.$$

To compute the rest summands, we use Lyndon-Hochschild-Serre cohomology spectral sequence with coefficient $\mathbb{Z}[1/p]$ as follows

$$E^2_{p,q} = H_p\left(D, H_q(U,\mathbb{Z}[1/p])\right) \Rightarrow H_{p+q}(B,\mathbb{Z}[1/p]) .$$

Lemma 8 *Given G is a finite group and the ring $\mathbb{Z}[1/p]$ as a trivial $G-$module. Then for $n > 0$, $H_n(G,\mathbb{Z}[1/p]) \cong \oplus_{q\neq p} H_n(G,\mathbb{Z})_{(q)}$. In other words, the coefficient $\mathbb{Z}[1/p]$ kills the $p-$primary part in the integral homology of G.*

Proof. Using Universal Coefficient Theorem,

$$H_n(G,\mathbb{Z}[1/p]) = H_n(G,\mathbb{Z}) \otimes_{\mathbb{Z}\mathbb{G}} \mathbb{Z}[1/p] \oplus \mathrm{Tor}_{ZG}\left(H_{n-1}(G,\mathbb{Z}),\mathbb{Z}[1/p]\right).$$

Obviously, $Tor\left(H_{n-1}(G,\mathbb{Z}),\mathbb{Z}[1/p]\right) = 0$ since $\mathbb{Z}[1/p]$ is torsion-free. Also tensoring with $\mathbb{Z}[1/p]$ kills the $p-$primary part of $H_{n-1}(B,\mathbb{Z})$ since if $p^r x = 0$ then

$$x \otimes y = x \otimes \left(p^r \frac{y}{p^r}\right) = p^r x \otimes \frac{y}{p^r} = 0.$$

Moreover, if $q^r x = 0$ for some q prime to p then there exist $a, b \in \mathbb{Z}$ such that $aq^r + bp^k = 1$. Thus

$$x \otimes \frac{m}{p^k} = x\left(aq^r + bp^k\right) \times \frac{m}{p^k} = bm \otimes 1.$$

Therefore, $H_n(G,\mathbb{Z}[1/p]) \cong \oplus_{q\neq p} H_n(G,\mathbb{Z})_{(q)}$.

Now consider the ring $\mathbb{Z}[1/p]$ as a trivial $B-$module. Then the ring $\mathbb{Z}[1/p]$ can be considered as a trivial $U-$module (a trivial $T-$module). Thus, the Lemma 8 gives us the following theorem.

Theorem 9 $H_{t+s}(B,\mathbb{Z}[1/p]) = E^2_{t,s} = \begin{cases} \mathbb{Z}[1/p] & \text{if } s = t = 0 \\ \mathbb{Z}_{p-1} & \text{if s = 0 and todd} \\ 0 & \text{otherwise} \end{cases}.$

Proof. $H_s(U,\mathbb{Z}[1/p]) = 0$ for all $s > 0$ and $H_0(U,\mathbb{Z}[1/p]) \cong \mathbb{Z}[1/p]$. Since only $H_0(U,\mathbb{Z}[1/p])$ is non-zero and the group T acts trivially on $H_0(U,\mathbb{Z}[1/p])$ we have $E^2_{t,0} = H_t\left(\mathbb{F}_p^*,\mathbb{Z}[1/p]\right) \cong \mathbb{Z}/(p-1)$ for t odd. Obviously, $E^2 = E^\infty \Rightarrow H_n(B,\mathbb{Z}[1/p])$.

Lemma 8 also gives us that

$$H_n(B,\mathbb{Z}[1/p]) = \oplus_{q\neq p} H_n(B,\mathbb{Z})_{(q)}.$$

Hence, $H_n(B,\mathbb{Z}) = H_n(B,\mathbb{Z})_{(p)} \oplus H_n(B,\mathbb{Z}[1/p])$. In conclusion, one gets the following theorem.

Theorem 10 *For $p \geq 5$. Then*

$$H_n(B,\mathbb{Z}) = \begin{cases} \mathbb{Z}/(p-1) & \text{if n is odd and n } \not\equiv (p-2) \\ \mathbb{Z}/(p-1) \oplus \mathbb{Z}/p & \text{if } n \equiv 0\;(p-2) \\ 0 & \text{otherwise} \end{cases}.$$

COMPETING INTERESTS

The authors declare that they have no conflicts of interest.

AUTHOR CONTRIBUTION

Vo Quoc Bao have contributed the presentation and the periodicity of Borel subgroup of SL(2,Fp) and have written the manuscript. Bui Anh Tuan have contributed the integral homology of Borel subgroups B of SL(2,Fp) and revising the manuscript.

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