ALMOST ε-QUASISOLUTIONS OF A NONCONVEX PROGRAMMING PROBLEM WITH AN INFINITE NUMBER OF CONSTRAINTS

Tran Van Thach(1), **Ta Quang Son**(2)

(1) Thudaumot University

(2) Nhatrang College of Education

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ABSTRACT: *Under a generalized Karush-Kuhn-Tucker condition up to ε, we establish some sufficient optimality conditions for almost ε-quasisolutions of a nonconvex programming problem which has an infinite number of constraints. Some results on ε-weak duality in Mond-Weir type for the problem are also introduced.*

Keywords: *Generalized Karush-Kuhn-Tucker condition up to ε, almost ε-quasisolution, ε-duality in Mond-Weir type.*

INTRODUCTION

In this paper, we establish some approximate optimality conditions for a nonconvex optimization problem. This topic has attracted many authors for years [10], [7], [14], [9], [11], [2], [3], [13], [14]. In optimization, characterizing approximate solutions of a problem is essential since, numerically, only approximate solutions can be obtained [13]. Besides concept of *ε-solutions* which has global character, there were concepts of approximate solutions which have local ones, such as *ε-quasisolutions, almost ε-quasisolutions* (Definition 2.3). The global character of ε-solutions is suitable for convex optimzation problems. The local one of *ε-quasisolutions* or *almost ε-quasisolutions,* meanwhile, is useful for nonconvex optimization problems.

We deal with sufficient optimality conditions for almost ε-quasisolutions of a nonconvex optimization problem formulated as follows:

$$
\begin{aligned}
(\mathrm{P}) \quad \text{Minimize} \quad & f(x) \\
s.t. \quad & f_t(x) \le 0, t \in T, \\
& x \in C,
\end{aligned}
$$

where $f, f_t : X \to R, t \in T$, are locally Lipschitz functions on a Banach space X, T is an index set not necessarily finite, C is a nonempty closed subset of X. Our results develope some ones from the paper [13], where approximate sufficient optimality conditions were established under a Karush-Kuhn-Tucker (KKT) condition and the properties of regularity and semiconvexity applied for functions involved. To start with it, we reconsider necessary/sufficient conditions for approximate optimality solutions for (P). Then, by relaxing or modifying conditions on the data for (P), we establish some new versions of approximate sufficient optimality conditions for (P). In our new results, we use property ε-semiconvexity for locally Lipschitz functions instead of property regularity or semiconvexity. Note that, for the problem (P), there were some results on ε-weak duality of the problem in Wolfe type presented in [13]. Here we give some results of ε-weak duality of (P) in Mond-Weir type.

The paper is organized as follows: The next section is devoted to preliminaries. Definitions of semiconvex functions and ε-semiconvex functions are recalled in this part. We also recall some definitions of local approximate solutions such as ε-quasisolutions, almost ε-quasisolutions. Our main results are in the last section. Several versions of optimality theorems for ε-quasisolutions of (P) are established. Some results on ε-duality of (P) in Mond-Weir type are given in this section.

PRELIMINARIES

Throughout this paper, X is a Banach space, T is a compact topological space, $f: X \to R$ is a locally Lipschitz function, $f_t : X \to R,\ t \in T$, are locally Lipschitz with respect to x uniformly in t, i.e., that for each $x \in X$, there exists a neighborhood U of x and a constant $K > 0$ such that $|f_t(z) - f_t(z')| \le K \| z - z' \|, \forall z, z' \in U, \forall t \in T.$ The following concepts can be found in the Clarke's book [1]. Let $g : X \to R$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$, denoted by $g'(z;d)$, is defined by

$$g'(z;d) = \lim_{t \to 0} \frac{g(z+td) - g(z)}{t}$$

if the limit exists. The Clarke generalized directional derivative of g at $z \in X$ in direction $d \in X$, denoted by $g^c(z;d)$, and the Clarke subdifferential of g at $z \in X$, denoted by $\partial^c g(z)$, are defined by

$$g^c(z;d) = \lim_{\substack{x \to z \\ t \downarrow 0}} \frac{g(x+td) - g(x)}{t} \text{ and } \partial^c g(z) = \left\{ v \in X^* \mid v(d) \le g^c(z;d), \forall d \in X \right\},$$

respectively.

A locally Lipschitz function g is said to be quasidifferentiable (or regular in the sense of Clarke) at $z \in X$ if $g'(z;d)$ exists and $g^c(z;d) = g'(z;d)$ for all $d \in X$.

Let D be a nonempty closed subset of X. Let X^* be the dual space of X. The normal cone to $z \in D$ is defined by:

$N(D,z)=\{u\in X^* \mid u(v)\le 0, \forall v\in T_D(z)\}$

where $T_D(z)=\{v\in X \mid d_D^c(z;v)=0\}$ denotes the tangent cone to D at z, and d_D^c is the distance function to D. When D is convex, $N(D,z)$ coincides with the normal cone in the sense of convex analysis:

$N(D,z)=\{u\in X^* \mid u(x-z)\le 0, \forall x\in D\}$.

Definition 2.1 [8] Let C be a subset of X. A function $g:X\to R$ is said to be semiconvex at $z\in C$ if the following statements are satisfied:

(i) g is locally Lipschitz at z,

(ii) g is regular at z, and

(iii) $d\in X, z+d\in C$, $g'(z;d)\ge 0 \Rightarrow g(z+d)\ge g(z)$.

The function g is said to be semiconvex on C if g is semiconvex at every $z\in C$.

Lemma 2.1 [8, Theorem 8] Let C be a convex subset of X. If $g:X\to R$ is a semiconvex function on C then

$z\in C, d\in X, z+d\in C,\ g(z+d)\le g(z) \Rightarrow g'(z;d)\le 0.$

The following definition is extended from Definition 2.1.

Definition 2.2 [13] Let $\varepsilon\ge 0$. A function $g:X\to R$ is said to be ε-semiconvex at $z\in C$ if the hypothesis (i) and (ii) of Definition 2.1 hold and

$$d\in X, z+d\in C, g'(z;d)+\sqrt{\varepsilon}\,\|d\|\ge 0 \Rightarrow g(x)+\sqrt{\varepsilon}\,\|d\|\ge g(z). \tag{2.1}$$

The function g is said to be ε-semiconvex on C if g is ε-semiconvex at every $z\in C$.

We use the following linear space:

$R^{(T)} := \{(\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t\in T \text{ but only finitely many } \lambda_t \ne 0\}$. With $\lambda=(\lambda_t)\in R^{(T)}$ the supporting set coressponding to λ is $T(\lambda)=\{t\in T \mid \lambda_t\ne 0\}$. Obviously, it is a finite subset of T. The nonnegative cone of $R^{(T)}$ is defined by,

$R_+^{(T)}=\{(\lambda_t)\in R^{(T)} \mid \lambda_t\ge 0, \forall t\in T\}$.

It is easy to see that this cone is convex. The space $R^{(T)}$ can be equiped a norm formulated as follows

$$\|\lambda\|_1 := \sum_{t\in T} |\lambda_t| = \sum_{t\in T(\lambda)} |\lambda_t|.$$

With $\lambda \in R_+^{(T)}$ and $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space,

$$\sum_{t\in T} \lambda_t z_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if } T(\lambda) \neq \varnothing, \\ 0 & \text{if } T(\lambda) = \varnothing. \end{cases}$$

For $f_t, t \in T$, and for $\{Y_t\}_{t\in T}$, a family of nonempty subsets of X:

$$\sum_{t\in T} \lambda_t f_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t f_t & \text{if } T(\lambda) \neq \varnothing, \\ 0 & \text{if } T(\lambda) = \varnothing, \end{cases}$$

and

$$\sum_{t\in T} \lambda_t Y_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t Y_t & \text{if } T(\lambda) \neq \varnothing, \\ 0 & \text{if } T(\lambda) = \varnothing. \end{cases}$$

For the problem (P), we denote by A the feasible set of (P). Let $\varepsilon \geq 0$. The ε-feasible set of (P) is

$$A_\varepsilon := \left\{ x \in C \mid f_t(x) \leq \sqrt{\varepsilon}, \forall t \in T \right\}.$$

Definition 2.3 *Let $\varepsilon \geq 0$. A point $z_\varepsilon \in X$ is said to be*

(i) An almost ε-solution of (P) *if $z_\varepsilon \in A_\varepsilon$ and $f(z_\varepsilon) \leq f(x) + \varepsilon$ for all $x \in A$;*

(ii) An almost ε-quasisolution of (P) *if $z_\varepsilon \in A_\varepsilon$ and $f(z_\varepsilon) \leq f(x) + \varepsilon \| x - z_\varepsilon \|$ for all $x \in A$;*

(iii) An almost regular ε-solution of (P) *if $z_\varepsilon \in A_\varepsilon$ is an almost ε-solution and is an almost ε-quasisolution of* (P).

When $z_\varepsilon \in A$, we obtain concepts ε-solution, ε-quasisolution, and regular ε-solution of (P), *respectively.*

RESULTS AND DISCUSSION

Approximate optimality conditions

To establish sufficient conditions for almost ε-quasisolution of (P), we recall some results presented in [13]. Let us denote by ($\mathcal{A}$) the fact that at least one of the following conditions is satisfied:

(a1) X is separable;

(a2) X is metrizable and $\partial^c f_t(x)$ is upper semicontinuous in $t \in T$ for every $x \in X$.

Let $z_\varepsilon \in A$ and let $I(z_\varepsilon) = \{t \in T \mid f_t(z_\varepsilon) = 0\}$. We denote by ($\mathcal{U}$) the following constraint qualification condition:

$\exists d \in T_C(z_\varepsilon) : f_t^c(z_\varepsilon; d) < 0, \forall t \in I(z_\varepsilon) = \{t \in T \mid f_t(z_\varepsilon) = 0\}$. **Proposition 3.1** [13, Theorem 4.1] Let $\varepsilon \geq 0$ and let $z_\varepsilon \in A$ be an ε-quasisolution of (P). Suppose that the condition ($\mathcal{A}$) is satisfied. If the constraint qualification condition ($\mathcal{U}$) holds and the convex hull of $\{\bigcup \partial^c f_t(z_\varepsilon) \mid t \in T_C(z_\varepsilon)\}$ is weak *-closed then there exists $\lambda \in R_+^{(T)}$ such that

$$0 \in \partial^c f(z_\varepsilon) + \sum_{t \in T} \lambda_t \partial^c f_t(z_\varepsilon) + N(C, z_\varepsilon) + \sqrt{\varepsilon} B^*,\ f_t(z_\varepsilon) = 0, \forall t \in T(\lambda) \qquad (3.1)$$

where B^* is a closed unit ball in X^*.

If a pair (z_ε, λ) satisfies the condition (3.1) then it is called a Karush-Kuhn-Tucker (KKT) pair up to ε. The following definition is an extension of it.

Definition 3.1 [13, Definition 4.1] Let $\varepsilon \geq 0$. A pair $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ is called a generalized KKT condition up to ε if

$0 \in \partial^c f(z_\varepsilon) + \sum_{t \in T} \lambda_t \partial^c f_t(z_\varepsilon) + N(C, z_\varepsilon) + \sqrt{\varepsilon} B^*,\ f_t(z_\varepsilon) \geq 0, \forall t \in T(\lambda)$ where B^* is a closed unit ball in X^*. It is called strict if $f_t(z_\varepsilon) > 0$ for all $t \in T(\lambda)$, which is equivalent to $\lambda_t = 0$ if $f_t(z_\varepsilon) \leq 0$.

The definition above is reasonable since it was shown in [13] that there exist z_ε is an almost ε-quasisolution of (P) and $\lambda \in R_+^{(T)}$ such that (z_ε, λ) is a generalized KKT pair up to ε. So, the such generalized KKT pair up to ε can be used as a hypothesis to survey the existence of almost ε-quasisolutions of (P).

Theorem 3.1 [13, Theorem 4.3] For the problem (P), assume that C is convex and that the functions f_t, $t \in T$, are convex. Let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ be a generalized KKT pair up to ε. If f is ε-semiconvex at z_ε with respect to C, then

$f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon} \| x - z_\varepsilon \|$ for all $x \in C$ such that

$f_t(x) \le f_t(z_\varepsilon),\ \forall t \in T(\lambda)$.

In particular, z_ε is an almost ε-quasisolution for (P).

By modifying or changing assumptions applied to the involved functions of (P), we will give some new versions of the theorem above.

Theorem 3.2 For the problem (P), let $\varepsilon \ge 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ be a generalized KKT pair up to ε. Suppose that C is convex, the function f is ε-semiconvex at z_ε, and the functions f_t, $t \in T$, are semiconvex at z_ε. Then

$f(z_\varepsilon) \le f(x) + \sqrt{\varepsilon}\, \| x - z_\varepsilon \|$ for all $x \in C$ such that

$f_t(x) \le f_t(z_\varepsilon),\ \forall t \in T(\lambda)$.

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. Let $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ be a generalized KKT pair up to ε of (P). We have

$0 \in \partial^c f(z_\varepsilon) + \sum_{t\in T} \lambda_t \partial^c f_t(z_\varepsilon) + N(C, z_\varepsilon) + \sqrt{\varepsilon} B^*,\ f_t(z_\varepsilon) \ge 0, \forall t \in T(\lambda)$. If $T(\lambda) = \varnothing$ then $\lambda_t = 0$ for all $t \in T$. We get

$$0 \in \partial^c f(z_\varepsilon) + N(C, z_\varepsilon) + \sqrt{\varepsilon} B^*,\ f_t(z_\varepsilon) \ge 0, \forall t \in T(\lambda).$$

Hence, there exist $u \in \partial^c f(z_\varepsilon),\ w \in N(C, z_\varepsilon), v \in B^*$ such that

$$u(x - z_\varepsilon) + \sqrt{\varepsilon}\, \| x - z_\varepsilon \| \ge u(x - z_\varepsilon) + \sqrt{\varepsilon} v(x - z_\varepsilon) = -w(x - z_\varepsilon) \ge 0, \forall x \in C.$$

So, $f'(z_\varepsilon; x - z_\varepsilon) + \sqrt{\varepsilon}\, \| x - z_\varepsilon \| \ge f(z_\varepsilon), \forall x \in C$. Since f is ε-semiconvex at z_ε,

$$f(x) + \sqrt{\varepsilon}\, \| x - z_\varepsilon \| \ge f(z_\varepsilon), \forall x \in C.$$

Noting that $A \subset C$, the inequality above holds for all $x \in A$. Since $z_\varepsilon \in A_\varepsilon$, z_ε is an almost ε-quasisolution for (P).

If $T(\lambda) \ne \varnothing$ then $f_t(z_\varepsilon) \ge 0$ for all $t \in T(\lambda)$. Since $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ is a generalized KKT pair, there exist $u \in \partial^c f(z_\varepsilon),\ u_t \in \partial^c f_t(z_\varepsilon), t \in T,\ w \in N(C, z_\varepsilon),\ v \in B^*$ such that $u + \sum_{t\in T} \lambda_t u_t + \sqrt{\varepsilon} v = -w$. Hence,

$$u(x-z_\varepsilon)+\sum_{t\in T}\lambda_t u_t(x-z_\varepsilon)+\sqrt{\varepsilon}v(x-z_\varepsilon)=-w(x-z_\varepsilon)\geq 0,\forall x\in C. \quad (3.2)$$

Since $f_t, t\in T$, are semiconvex at z_ε and $f_t(x)\leq f_t(z_\varepsilon)$ for all $t\in T(\lambda)$,

$$u_t(x-z_\varepsilon)\leq f^c(z_\varepsilon;x-z_\varepsilon)=f^{'}(z_\varepsilon;x-z_\varepsilon),\ \forall t\in T(\lambda),\forall x\in C.$$

Then, from (3.2), we obtain $u(x-z_\varepsilon)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq 0,\forall x\in C$. Using a similar argument as above, we deduce that

$$f(x)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq f(z_\varepsilon),\forall x\in C,$$

and z_ε is an almost ε-quasisolution for (P). □

Remark 3.1 Since a convex function is a semiconvex function (see [6], [13]), we can see that Theorem 3.1 ([13, Theorem 4.3]) is a consequence of Theorem 3.2.

In the next theorem, assumptions applied for the constraint functions are relaxed. Concretely, we use the regularity property of $f_t, t\in T$, instead of the semiconvexity. We need the Lagrange function associated to (P):

$$L(x,\lambda)=\begin{cases} f(x)+\sum_{t\in T}\lambda_t f_t(x), & (x,\lambda)\in C\times\mathbb{R}_+^{(T)},\\ +\infty, & (x,\lambda)\notin C\times\mathbb{R}_+^{(T)}.\end{cases}$$

Note that sum of two ε-semiconvex (semiconvex) functions may not be an ε-semiconvex (semiconvex) function. We propose a following theorem.

Theorem 3.3 For the problem (P), let $\varepsilon\geq 0$ and let $(z_\varepsilon,\lambda)\in A_\varepsilon\times R_+^{(T)}$ be a generalized KKT pair up to ε. Suppose that C is convex, the functions f and f_t, $t\in T$, are regular at z_ε. If $L(\cdot,\lambda)$ is ε-semiconvex at z_ε then

$f(z_\varepsilon)\leq f(x)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|$ for all $x\in C$ such that

$f_t(x)\leq f_t(z_\varepsilon),\ \forall t\in T(\lambda)$. In particular, z_ε is an almost ε-quasisolution for (P).

Proof. Let $(z_\varepsilon,\lambda)\in A_\varepsilon\times R_+^{(T)}$ be a generalized KKT pair up to ε. If $T(\lambda)=\varnothing$, the proof is similar to the coressponding case in the proof of Theorem 3.2. If $T(\lambda)\neq\varnothing$ then $f_t(z_\varepsilon)\geq 0$ for all $t\in T(\lambda)$. The proof is the same as that of Theorem 3.2. There exists $u\in\partial^c f(z_\varepsilon)$, $u_t\in\partial^c f_t(z_\varepsilon), t\in T$, $w\in N(C,z_\varepsilon)$, $v\in B^*$ such that

$$u(x-z_\varepsilon)+\sum_{t\in T}\lambda_t u_t(x-z_\varepsilon)+\sqrt{\varepsilon}v(x-z_\varepsilon)=-w(x-z_\varepsilon)\geq 0,\forall x\in C:$$

Since $f_t, t\in T$, and f are regular at z_ε, we get

$$f^{'}(z_\varepsilon,x-z_\varepsilon)+\sum_{t\in T}\lambda_t f_t^{'}(z_\varepsilon,x-z_\varepsilon)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq 0,\forall x\in C.$$

It follows that $L^{'}(\cdot,\lambda)(z_\varepsilon;x-z_\varepsilon)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq 0$ for all $x\in C$. Since $L(\cdot,\lambda)$ is ε-semiconvex at z_ε, we obtain

$$f(x)+\sum_{t\in T}\lambda_t f_t(x)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq f(z_\varepsilon)+\sum_{t\in T}\lambda_t f_t(z_\varepsilon),\forall x\in C.$$

On the other hand, under assumption that $f_t(x)\leq f_t(z_\varepsilon)$ for all $t\in T(\lambda)$ and for all $x\in C$, we deduce that

$$f(x)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|\geq f(z_\varepsilon),\forall x\in C.$$

Since $A\subset C$, z_ε is an almost ε-quasisolution for (P). □

The following example shows that there exists the ε-feasible set is a convex but the constraint functions may not ε-semiconvex.

Example: Let $A_\varepsilon=\left\{x\in C\,|\,x^3\leq\sqrt{\varepsilon}\right\}$ where $C=[-1,1]$ and $\varepsilon=1/16$. A simple computation gives $A_\varepsilon=[-1,1/2]$, a convex set. We can check that the function $g(x)=x^3$ is not $1/16$-semiconvex at $z_\varepsilon=0$. Indeed, $g'(0;d)+\sqrt{\varepsilon}\,|d|=\sqrt{\varepsilon}\,|d|\geq 0$ for all $d\in R$. Choose $d=-1$. We get $z_\varepsilon+d\in A_\varepsilon$ and $g(0+d)+\sqrt{\varepsilon}\,|d|=-1+1/4=-3/4<g(0)=0$.

We now give a modified version of Theorem 3.3 by assuming that the ε-feasible set of (P), A_ε, is a convex subset of X.

Theorem 3.4 For the problem (P), let $\varepsilon\geq 0$ and let $(z_\varepsilon,\lambda)\in A_\varepsilon\times R_+^{(T)}$ be a generalized KKT pair up to ε. Suppose that C is convex, f_t, $t\in T$, are regular at z_ε, A_ε is a convex subset of X, and f is ε-semiconvex at z_ε then

$f(z_\varepsilon)\leq f(x)+\sqrt{\varepsilon}\,\|x-z_\varepsilon\|$ for all $x\in A_\varepsilon$ such that

$f_t(x)\leq f_t(z_\varepsilon),\ \forall t\in T(\lambda).$

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. Let $\varepsilon \geq 0$. Suppose that $(z_\varepsilon, \lambda) \in A_\varepsilon \times R_+^{(T)}$ is a generalized KKT pair up to ε of (P), the functions f_t, $t \in T$, are regular at z_ε, and the function f is ε-semiconvex at z_ε. If $T(\lambda) = \varnothing$, we use the argument as in the proof of Theorem 3.2. When $T(\lambda) \neq \varnothing$, similarly to the proof of Theorem 3.3 we obtain $u \in \partial^c f(z_\varepsilon)$, $u_t \in \partial^c f_t(z_\varepsilon), t \in T$, $w \in N(C, z_\varepsilon)$, $v \in B^*$ such that

$$u(x - z_\varepsilon) + \sum_{t\in T} \lambda_t u_t(x - z_\varepsilon) + \sqrt{\varepsilon}\, v(x - z_\varepsilon) = -w(x - z_\varepsilon) \geq 0, \forall x \in C. \qquad (3.3)$$

Since A_ε is a convex set and $z_\varepsilon \in A_\varepsilon$, for every $x \in A_\varepsilon$, we have

$$z_\varepsilon + \mu(x - z_\varepsilon) = \mu x + (1-\mu) z_\varepsilon \in A_\varepsilon, \forall \mu \in (0,1).$$

Hence, $f_t(z_\varepsilon + \mu(x - z_\varepsilon) - f_t(z_\varepsilon) \leq 0, \forall t \in T(\lambda), \forall \mu \in (0,1)$. Thus,

$$f_t'(z_\varepsilon; x - z_\varepsilon) = \lim_{\mu \to 0^+} \frac{f(z_\varepsilon + \mu(x - z_\varepsilon) - f(z_\varepsilon)}{\mu} \leq 0, \forall t \in T(\lambda).$$

Since $u_t \in \partial^c f_t(z_\varepsilon)$ and $f_t, t \in T$, are regular at z_ε,

$$u_t(x - z_\varepsilon) \leq f^c(z_\varepsilon; x - z_\varepsilon) = f'(z_\varepsilon; x - z_\varepsilon) \leq 0, \forall t \in T(\lambda), \forall x \in A_\varepsilon$$

Combining this and (3.3), we deduce that $u(x - z_\varepsilon) + \sqrt{\varepsilon} \| x - z_\varepsilon \| \geq 0$ for all $x \in A_\varepsilon$. Since f is ε-semiconvex, the desired conclusion follows. □

3.2 Approximate duality

The results on ε-weak duality of (P) in Wolfe type was presented in [13]. The last part of this paper is devoted to ε-weak duality of (P) in Mond-Weir type. Frequently, the dual problem of (P) in this type is formulated as follows:

$$\begin{aligned}
(\text{MD}) \quad & \text{Maximize} \quad f(y) \\
& s.t. \quad 0 \in \partial^c f(y) + \sum_{t\in T} \lambda_t \partial^c f_t(y) + N(C, y) + \sqrt{\varepsilon} B^*, \\
& \qquad\quad \lambda_t f_t(y) \geq 0,\ t \in T, \\
& \qquad\quad (y, \lambda) \in C \times R_+^{(T)}.
\end{aligned}$$

Let us denote by F the feasible set of (MD).

Theorem 3.5 *For the problem* (P), *suppose that C is convex, f is ε-semiconvex on C, and $f_t, t \in T$, are semiconvex on C. Then ε-weak duality between* (P) *and* (MD) *holds, i.e.,*

$$f(x)+\sqrt{\varepsilon}\,\|x-y\|\geq f(y),\forall x\in A,\forall(y,\lambda)\in F. \qquad (3.4)$$

Proof. Let x and (y,λ) be the feasible points of (P) and (MD), respectively. Since $0\in\partial^c f(y)+\sum_{t\in T}\lambda_t\partial^c f_t(y)+N(C,y)+\sqrt{\varepsilon}B^*$, there exist $u\in\partial^c f(y), u_t\in\partial^c f_t(y), t\in T, v\in B^*$ and $w\in N(C,y)$ such that

$$u(x-y)+\sum_{t\in T}\lambda_t u_t(x-y)+\sqrt{\varepsilon}v(x-y)=-w(x-y)\geq 0,\forall x\in C. \qquad (3.5)$$

As $x\in A$, we have $f_t(x)\leq 0$ for all $t\in T$. Since $\lambda_t f_t(y)\geq 0$ for all $t\in T$, it derives $f_t(x)\leq f_t(y)$ for all $t\in T(\lambda)$. Using a property of semiconvex function, we deduce that $u_t(x-y)\leq f_t'(y;x-y)\leq 0$ for all $t\in T(\lambda)$. So, this and (3.5) imply that

$$u(x-y)+\sqrt{\varepsilon}\,\|x-y\|\geq 0.$$

The conlusion follows since f is ε-semiconvex on C. □

Theorem 3.6 *For the problem* (P)*, suppose that C is convex,* $f, f_t, t\in T$*, are regular on C, and for each* $\lambda\in R_+^{(T)}$*,* $L(\cdot,\lambda)$ *is ε-semiconvex on C. Then ε-weak duality between* (P) *and* (MD) *holds.*

Proof. Let x and (y,λ) be the feasible points of (P) and (MD), respectively. Using a similar argument of the proof of the theorem above, there exist $u\in\partial^c f(y), u_t\in\partial^c f_t(y), t\in T$ such that

$$u(x-y)+\sum_{t\in T}\lambda_t u_t(x-y)+\sqrt{\varepsilon}\,\|x-y\|\geq 0,\forall x\in C.$$

Since $f, f_t, t\in T$, are regular on C, it follows that

$$L'(\cdot,\lambda)(y;x-y)+\sqrt{\varepsilon}\,\|x-y\|\geq 0,\forall x\in C.$$

Since $L(\cdot,\lambda)$ is ε-semiconvex on C then $L(x,\lambda)+\sqrt{\varepsilon}\,\|x-y\|\geq L(y,\lambda)$, *i.e.*,

$$f(x)+\sum_{t\in T}\lambda_t f_t(x)+\sqrt{\varepsilon}\,\|x-y\|\geq f(y)+\sum_{t\in T}\lambda_t f_t(y).$$

The desire result follows by $f_t(x)\leq 0$ and $\lambda_t f_t(y)\geq 0$ for all $t\in T$. □

Remark. *In the two theorems above, as* $x\in A_\varepsilon$*, the inequality* (3.4) *holds if we assume that* $f_t(x)\leq f_t(y)$ *for all* $t\in T(\lambda)$.

The following corollary is a consequence of the previous theorems.

Corollary 3.1 *Assume that at least one of the following statements are satisfied:*

a) f is ε-semiconvex on C and $f_t, t \in T$*, are semiconvex on C;*

b) $f, f_t, t \in T$*, are regular on C and for every* $\lambda \in R_+^{(T)}$*,* $L(\cdot,\lambda)$ *is ε-semiconvex on C;*

Then, for every feasible point (z,λ) *of* (MD),

(i) if $z \in A_\varepsilon$ *then z is an almost ε-quasisolution of* (P);

(ii) if $z \in A$ *then z is an ε-quasisolution of* (P).

ĐIỀU KIỆN TỐI ƯU CHO HẦU TỰA ε -NGHIỆM CỦA BÀI TOÁN TỐI ƯU KHÔNG LỒI CÓ VÔ HẠN RÀNG BUỘC

Trần Văn Thạch[(1)], Tạ Quang Sơn[(2)]

(1) Trường Đại học Thủ Dầu Một

(2) Trường Đại học Nha Trang

TÓM TẮT: *Dựa trên điều kiện Karush-Kuhn-Tucker suy rộng chính xác đến* ε*, chúng tôi thiết lập một số điều kiện đủ tối ưu cho các hầu tựa ε-nghiệm của bài toán qui hoạch không lồi có vô hạn ràng buộc. Một số kết quả về ε-đối ngẫu yếu dạng Mond-Weir cho bài toán cũng được giới thiệu.*

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