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Research Article

The rate of convergence in the central limit theorem via Zolotarev probability metric

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ABSTRACT

The central limit theorem is a well-known theorem in probability theory. It is the theoretical basis for constructing statistical problems such as parameter estimation problems and statistical hypothesis testing, etc. The main aim of this article is to estimate the upper bound in the central limit theorem for independent but not necessarily identically distributed random variables under Lyapunov's conditions via the Zolotarev probability metric. The obtained result is the rate of convergence in the central limit theorem for independent random variables. In the case of independent identically distributed random variables will be concluded as a direct corollary. The Zolotarev probability metric is the main research tool in this paper since it is an ideal metric of order s > 0. Furthermore, the Zolotarev probability metric may be compared with well-known metrics like the Kolmogorov metric, total variation metric, the Levy-Prokhorov metric, and the metric based on the Trotter operator, etc.

Key words: Central limit theorem, Zolotarev probability metric, Lyapunov's condition

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INTRODUCTION

The central limit theorem is one of the important theorems in the theory of probability and statistics, as a pearl in this field. So far, there are many research topics related to this theorem that has been attracting the attention of many mathematicians, such as proving and estimating the rate of convergence with various hypothetical conditions.

Probability metrics, moment method, operator method, and Stein method, etc., are tools and methods commonly used in the study of limit theorems. Each tool or method has its strengths. Realizing that the Zolotarev probability metric tool is easy for estimating the convergence rate in the central limit theorem, this metric has not been used before. Thus, we have decided to choose the Zolotarev probability metric as the main research tool in this paper.

Let $\{X_k\}$ be a sequence of independent, but not necessarily identically distributed random variables with $E|X_k|^{2+\delta} < +\infty$ for some $\delta > 0$ and $k \geq 1$. Let us put

$$m_k = E(X_k),\ \sigma_k^2 = Var(X_k),$$
$$B_n = \sum_{k=1}^{n} \sigma_k^2.$$

The following condition

$$\lim_{n\to+\infty} \left\{ B_n^{-1-\delta/2} \sum_{k=1}^{n} E|X_k - m_k|^{2+\delta} \right\} = 0$$

is called the Lyapunov's condition (see [1], Petrov 1995 – page 126).

In this paper, the upper bound of

$$d_s\left(B_n^{-1/2} \sum_{k=1}^{n} (X_k - m_k), Z \right)$$

will be estimated, where d_s is the Zoloatrev probability metric of order s (see Section 2 for more details) and Z $\sim$ Normal (0,1).

Throughout the article, the set of real numbers is denoted by $\mathbb{R} = (-\infty, +\infty)$ and the set of natural numbers is denoted by $\mathbb{N} = \{1, 2, ...\}$. The symbols $=^d$ and $\to^d$ express equality of distributions and convergence in distribution, respectively.

PRELIMINARIES

We denote by $\mathfrak{X}$ the set of random variables defined on the probability space (Ω, F, P) and by $C(\mathbb{R})$ the set of all real-valued, bounded, uniformly continuous functions defined on $\mathbb{R}$ with norm $||f|| = \sup_{x\in\mathbb{R}} |f(x)|$.

Moreover, for any $r \in \mathbb{N}$, $\delta \in (0, 1]$ and $s = r + \delta$, let us set

$$C^r(\mathbb{R}) = \{f \in C(\mathbb{R}) :\ f^{(k)} \in C(\mathbb{R}), 1 \leq k \leq r\}$$

and

$$D_s = \{f \in C^r(\mathbb{R}) :\ |f^{(r)}(x) - f^{(r)}(y)| \leq |x - y|^{\delta}\},$$

where $f^{(r)}$ is derivative function of order r of f.

The definition of the Zolotarev probability metric and its some basic properties will be recalled from [2–5] and [6] as follows.

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Definition. Let $X,Y \in \mathfrak{X}$. Zolotarev's probability metric on $\mathfrak{X}$ between two random variables X and Y is defined by

$$d_s(X,Y) = \sup_{f \in D_s} |E[f(X) - f(Y)]|.$$

The following properties of the Zolotarev probability metric will be used in the next section.

1. Zolotarev probability metric d_s is an ideal metric of order $s > 0$, i.e., for any $c \neq 0$ we have

$$d_s(cX, cY) = |c|^s d_s(X,Y),$$

and with V is independent of X and Y then

$$d_s(X+V, Y+V) \leq d_s(X,Y).$$

2. If $d_s(X_n, X_0) \longrightarrow 0$ as n→+∞ then $X_n \longrightarrow^d X_0$ as n→+∞.

The following lemma states one of the most crucial properties of the Zolotarev probability metric, used in the proof of the main result.

Lemma. Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of independent random variables (in each sequence) and they are independent. Then for all n $\in \mathbb{N}$,

$$d_s\left(\sum_{k=1}^n X_k, \sum_{k=1}^n Y_k\right) \leq \sum_{k=1}^n d_s(X_k, Y_k). \qquad (2.1)$$

Based on the ideality of the Zolotarev probability metric of order s>0, this lemma is easily proved by the mathematical inductive method. Hence its detailed proof is omitted.

MAIN RESULT

Before starting the main result, we prove an auxiliary result that will use in the subsequent theorem. According to [1] (Petrov 1995 — page 11), a random variable Y is said to have a normal distribution with parameters μ and σ^2, denoted by Y~Normal(μ, σ^2), if its characteristic function is given by

$$\varphi_Y(t) = exp\left\{i\mu t - \frac{\sigma^2}{2}t^2\right\}, \; t \in \mathbb{R}.$$

Proposition. Let $\{Y_k\}$ be a sequence of independent random variables and Y_k~Normal$(0, \sigma_k^2)$ for k ≥ 1. Let

$$T_n = \sum_{k=1}^n Y_k \text{ and } B_n = \sum_{k=1}^n \sigma_k^2.$$

Then

$$Z =^d B_n^{-1/2} T_n, \qquad (3.1)$$

where Z~Normal(0,1).

Proof. Since Y_k~Normal$(0, \sigma_k^2)$, their characteristic functions are given by

$$\varphi_{Y_k}(t) = e^{-\frac{\sigma_k^2}{2}t^2}$$

for $t \in \mathbb{R}$ and k ≥ 1. The characteristic function of T_n is defined by

$$\begin{aligned}\varphi_{T_n}(t) &= E(e^{iT_n t}) = E[e^{i(Y_1+Y_2+\ldots+Y_n)t}] \\ &= \varphi_{Y_1}(t).\varphi_{Y_2}(t)\ldots\varphi_{Y_n}(t) \\ &= e^{-\frac{\sigma_1^2}{2}t^2}.e^{-\frac{\sigma_2^2}{2}t^2}\ldots e^{-\frac{\sigma_n^2}{2}t^2} \\ &= e^{-\frac{\sigma_1^2+\sigma_2^2+\ldots+\sigma_n^2}{2}t^2} = e^{-\frac{B_n}{2}t^2}.\end{aligned}$$

Thus, the characteristic function of $B_n^{-1/2}T_n$ will be

$$\varphi_{B_n^{-1/2}T_n}(t) = \varphi_{T_n}\left(B_n^{-1/2}t\right) = e^{-\frac{t^2}{2}} = \varphi_Z(t).$$

This confirms that the representation (3.1) holds. The proof is finished.

Theorem. Let $\{X_k\}$ be a sequence of independent random variables with moments $E(X_k) = 0, E|X_k|^{2+\delta} < +\infty$ for $\delta \in (0,1]$ and k ≥ 1. Let

$$S_n = \sum_{k=1}^n X_k, \; \sigma_k^2 = Var(X_k) \text{ and } B_n = \sum_{k=1}^n \sigma_k^2.$$

Assume that $\{Y_k\}$ is a sequence of independent and normal distributed random variables with parameters 0 and σ_k^2, i.e., Y_k~Normal$(0, \sigma_k^2)$ for k ≥ 1. Moreover, suppose that the following Lyapunov's conditions are satisfied as n→+∞

$$\left\{B_n^{-1-\delta/2}\sum_{k=1}^n E|Y_k|^{2+\delta}\right\} = o(1)$$

and

$$\left\{B_n^{-1-\delta/2}\sum_{k=1}^n E|X_k|^{2+\delta}\right\} = o(1).$$

Then

$$d_{2+\delta}\left(B_n^{-1/2}S_n, Z\right) \leq \frac{1}{2}B_n^{-1-\delta/2}\sum_{k=1}^n\left(E|X_k|^{2+\delta} + E|Y_k|^{2+\delta}\right),$$

where Z~Normal(0,1).

Proof. According to the above Proposition,

$$Z =^d B_n^{-1/2}T_n.$$

Based on the ideality of the Zolotarev probability metric and from Lemma in the previous section, we have

$$\begin{aligned}&d_{2+\delta}\left(B_n^{-1/2}S_n, Z\right) \\ &= d_{2+\delta}\left(B_n^{-1/2}S_n, B_n^{-1/2}T_n\right) \\ &= B_n^{-1-\delta/2}d_{2+\delta}(S_n, T_n) \\ &\leq B_n^{-1-\delta/2}\sum_{k=1}^n d_{2+\delta}(X_k, Y_k).\end{aligned}$$

For any $f \in D_{2+\delta}$, $x \in \mathbb{R}$ and $\theta \in (0,1)$, by the Taylor series expansion with Lagrange remainder, we get

$$f(x) = f(0) + f'(0).x + \frac{f''(\theta x)}{2}x^2$$
$$= f(0) + f'(0).x + \frac{f''(0)}{2}x^2 + \frac{x^2}{2}[f''(\theta x) - f''(0)].$$

Note that for $k \geq 1$,

$$E(X_k) = E(Y_k) = 0 \text{ and}$$
$$E(X_k^2) = E(Y_k^2) = \sigma_k^2.$$

Hence

$$E[f(X_k) - f(Y_k)] =$$
$$E\{\frac{X_k^2}{2}[f''(\theta X_k) - f''(0)] - \frac{Y_k^2}{2}[f''(\theta Y_k) - f''(0)]\}.$$

Moreover, since $f \in D_{2+\delta}$ and $\theta \in (0,1)$, we can assert that

$$|E[f(X_k) - f(Y_k)]| =$$
$$\left|E\left\{\frac{X_k^2}{2}[f''(\theta X_k) - f''(0)] - \frac{Y_k^2}{2}[f''(\theta Y_k) - f''(0)]\right\}\right|$$
$$\leq E\left\{\frac{X_k^2}{2}|f''(\theta X_k) - f''(0)| + \frac{Y_k^2}{2}|f''(\theta Y_k) - f''(0)|\right\}$$
$$\leq E\left\{\frac{X_k^2}{2}|\theta X_k|^{\delta} + \frac{Y_k^2}{2}|\theta Y_k|^{\delta}\right\}$$
$$\leq \frac{1}{2}\left(E|X_k|^{2+\delta} + E|Y_k|^{2+\delta}\right).$$

By the definition of the Zolotarev probability metric, one has

$$d_{2+\delta}(X_k, Y_k)$$
$$= \sup_{f \in D_{2+\delta}} \{|E[f(X_k) - f(Y_k)]|\}$$
$$\leq \frac{1}{2}\left(E|X_k|^{2+\delta} + E|Y_k|^{2+\delta}\right).$$

Therefore

$$d_{2+\delta}\left(B_n^{-1/2}S_n, Z\right)$$
$$\leq \frac{1}{2}B_n^{-1-\delta/2}\sum_{k=1}^{n}\left(E|X_k|^{2+\delta} + E|Y_k|^{2+\delta}\right).$$

The proof is finished.

Remark. Assume that the hypothetical conditions of the above theorem are accepted. Then, the central limit theorem will establish as follows

$$B_n^{-1/2}S_n \to^d Z \sim \text{Normal}(0,1) \quad as \quad n \to +\infty.$$

It is worth saying that if $\{X_k\}$ is a sequence of independent identically distributed random variables with finite common moment $E(X_1) = m$, common variance $Var(X_1) = \sigma^2 \in (0, +\infty)$ and $E|X_1 - \mu|^{2+\delta} < +\infty$ for some $\delta > 0$ then the Lyapunov's condition is satisfied. Indeed, we now get

$$B_n = \sum_{k=1}^{n} \sigma_k^2 = n\sigma^2 \text{ and}$$
$$\sum_{k=1}^{n} E|X_k - m_k|^{2+\delta} = nE|X_1 - m|^{2+\delta}$$

and the Lyapunov's condition becomes

$$\lim_{n \to +\infty}\left\{\left(n\sigma^2\right)^{-1-\delta/2} nE|X_1 - m|^{2+\delta}\right\} = 0.$$

Therefore, the following result considers the case of a sequence of independent identically distributed random variables without Lyapunov's condition.

Corollary. Let $\{X_k\}$ be a sequence of independent identically distributed random variables with common moments $E(X_1) = 0, E(X_1^2) = \sigma^2 \in (0, +\infty)$ and $E|X_1|^{2+\delta} < +\infty$ for $\delta \in (0,1]$. Let Y~Normal(0, σ^2). Then

$$d_{2+\delta}\left(n^{-1/2}\sum_{k=1}^{n}\frac{X_k}{\sigma}, Z\right)$$
$$\leq \frac{n^{-\delta/2}}{2\sigma^{2+\delta}}\left(E|X_1|^{2+\delta} + E|Y|^{2+\delta}\right),$$

where Z~Normal(0, 1).

CONCLUDING REMARKS

In the same way as this study, the convergence rates in the weak limit theorems that their limit distributions are the stable laws can be established. Moreover, these issues may also extend to random sums.

It is worth saying that Lindeberg's condition is weaker than Lyapunov's one; that is, if Lyapunov's condition is satisfied, then Lindeberg's one is satisfied. Therefore, establishing the convergence rates in the limit theorems for independent random variables under Lindeberg's condition is also very interesting to investigate.

CONFLICT OF INTEREST

The authors confirm that there are no conflicts of interest in connection with this publication.

AUTHORS CONTRIBUTIONS

All authors contributed equally to this paper. The authors drafted the manuscript, read and approved the final version of the manuscript.

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