

OPTIMAL LIFTING WAVELET FILTER BANK DESIGN AND IMAGE COMPRESSION APPLICATION

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ABSTRACT: *The lifting scheme is an efficient tool to construct second-generation wavelets. It has been used to realize Daubechies wavelet transform in image compression standard JPEG-2000. Daubechies wavelets can provide better image coding performance than discrete cosine transform (DCT) which is used in JPEG because the wavelets can present signal more efficiently than DCT. However, for high compression rate, the details of the decompressed images in JPEG-2000 are degraded. The reason is that Daubechies filters are maximally flat while their frequency selectivity is very poor. In this paper, we present an efficient method for the optimal design of filter banks and wavelets based on the lifting structure. The design problem is expressed as an optimization problem where the frequency selectivity of filters is optimized for a given regularity order. The simulation results show that the filter banks designed by our proposed method can offer the coding performance improvement compared to Daubechies filters in JPEG-2000.*

Keywords: *Filter banks, wavelets, image coding, regularity, frequency selectivity, global optimization.*

1.INTRODUCTION

The discrete wavelet transform (DWT) has found in various signal processing applications, for example signal compression, denoising, watermarking, and so on, due to the fact that DWT can overcome the limitation of the traditional Fourier transform in being able to providing variable time and frequency resolutions [1]-[3]. As a result, the DWT has been adopted in international multimedia compression standards such as JPEG-2000 and MPEG4 [3], [4]. In the DWT based applications, proper choice of wavelets is critical to achieve systems with good performance. It is well-known that the wavelets can be generated by two-channel perfect reconstruction filter banks. As a result, the design of wavelets is equivalent to the design of perfect reconstruction two-channel filter banks. In addition to perfect reconstruction, the orthogonality and linear phase properties of the filter banks are desired in many applications. The orthogonal filter banks guarantee that the noise and error in subbands are not amplified, and hence, the coding system design is more simplified [3]. On the other hand, the linear phase is efficient for handling boundary distortions of finite length signals such as image signal. However, it is well-known that two-channel filter banks with both orthogonality and linear phase do not exist except for the Haar filters which are not continuous. Therefore, in practical applications orthogonality filter bank are often relaxed into bi-orthogonal filter banks.

In general, the filter bank design is a multi-objective optimization problem. The most important objective is perfect reconstruction, that is, the reconstructed signal is a delayed and scaled version of the original signal. Furthermore, additional properties of filters which are often required in certain applications are linear phase, flatness, high frequency selectivity. These design objectives are usually conflicting, and therefore, the design is required to have different tradeoffs. One class of popular filter banks with linear phase and maximally flatness

was introduced by Daubechies, for example, Daubechies 9-tap/7-tap filters with regularity order of 4 are used in JPEG-2000 [4], [5]. The maximally flat filters can be found in closed-form by Lagrange formula [6], [7]. However, it is well-known that maximally flat filters suffer from poor frequency selectivity. As a result, the image coding performance of maximally flat filters can be reduced for highly textured image. Therefore, the design of filter banks having optimal frequency characteristics for a given regularity order has been of great interest, see [9], [10] and references therein. However, the filter bank design is usually formulated as a highly nonlinear optimization due to the perfect reconstruction condition. Therefore, structures which are structurally imposed perfect reconstruction property are very attractive to simplify the design procedure.

An efficient filter bank structure satisfying perfect reconstruction is a lifting scheme. The lifting scheme of two-channel filter banks with two lifting steps was introduced by Phoong et al. [8]. This structure offers low implementation complexity and rich-features in filter frequency responses. However, only two extreme cases of filter frequency responses was considered. In the first case, the filters are designed by McClellan-Parks algorithm and Remez exchange algorithm. These algorithms result in the equi-ripple filters with lowest stopband attenuation without regularity. In the second case, the maximally flat filters with poor frequency selectivity was found by Lagrange formula. Consequently, these methods cannot allow to design the filters with arbitrary frequency responses and regularity orders.

In this paper, we propose a generalization method which can design the lifting scheme filter banks including filters with arbitrary frequency responses and regularity orders. For a prescribed regularity order, our design objective is to find an filter bank with the best frequency selectivity. We show that filter bank design can be formulated as a semi-definite programming problem whose globally optimal solutions can be efficiently solved by available softwares. One of advantages of our proposed method is that it can flexibly control the tradeoff between frequency selectivity and regularity. As a consequence, our filter bank can provide better image coding performance than the maximally flat filter banks for highly detailed images. The simulation results of our filter banks are presented to illustrate the performance of our proposed method. Moreover, the application of our filter bank in image coding is also presented to evaluate the effectiveness of our method.

The rest of the paper is organized as follows. In Section II, the lifting scheme of two-channel filter banks is briefly reviewed. The introduction of semi-definite programming is presented, and then the formulation for the two-channel filter bank design is derived in Section III. In Section IV, the design examples of the filter banks are given, and image coding performance of the filter banks is discussed. Finally, a concluding remarks are given in Section V.

Notations: Boldfaced lowercase letters are used to represent vectors, and boldfaced uppercase letters are reserved for matrices.

2. THE LIFTING SCHEME OF TWO-CHANNEL FILTER BANK

The lifting structures for the construction of bi-orthogonal wavelets are very efficient for the implementation because the analysis and synthesis filters can be jointly implemented. Moreover, the lifting schemes are robust to quantization noise, that means, the coefficient quantization does not affect the perfect reconstruction property [15], [16]. Therefore, the lifting structures have been attractive in practical applications. One lifting structure that can provide the filters with rich-features such as high frequency selectivity and regularity is a

halfband pair filter bank introduced by Phoong et al [8]. The halfband pair filter bank can be considered as the lifting scheme with two lifting steps, as shown in Fig. 1.

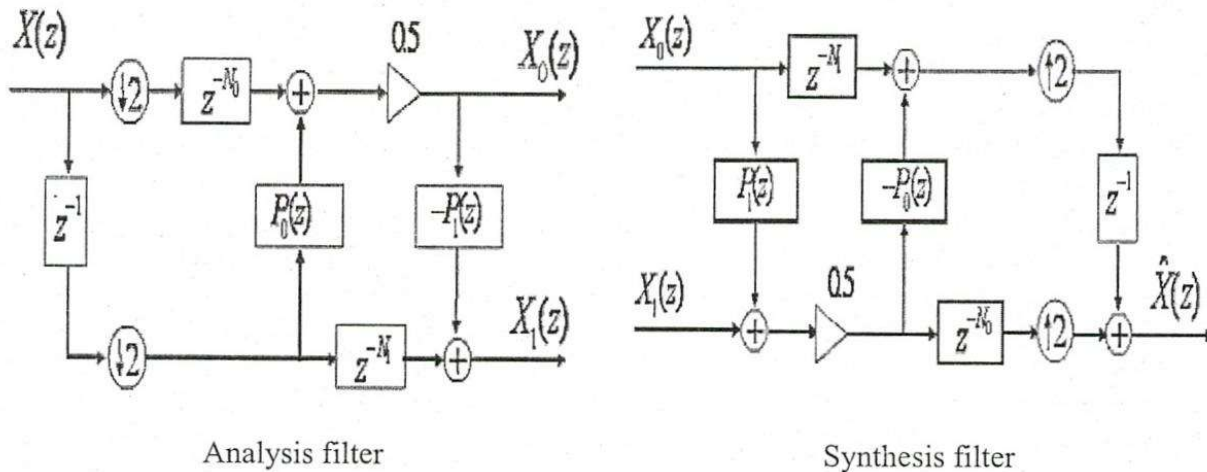


Fig.1. Lifting structure of filter bank with two lifting steps

The filter bank is parameterized by a subfilter pair $P_0(z)$ and $P_1(z)$. From Fig 1, the analysis filter are given by

$$H_0(z) = \frac{1}{2} (z^{-2N_0} + z^{-1}P_0(z^2)), \tag{1}$$

$$H_1(z) = z^{-(2N_1+1)} - P_1(z^2)H_0(z). \tag{2}$$

With the synthesis structure shown in Fig. 1, it can be verified that the filter bank is structurally perfect reconstruction for arbitrary choice of subfilters $P_0(z)$ and $P_1(z)$. The corresponding synthesis filters have the following form:

$$F_0(z) = -H_1(-z), \tag{3}$$

$$F_1(z) = H_0(-z). \tag{4}$$

By above relations, the synthesis filters $F_0(z)$, $F_1(z)$ are respectively lowpass and highpass filters if the analysis filters $H_0(z)$, $H_1(z)$ are lowpass and highpass ones. Therefore, the filter bank design reduces to finding a pair of subfilters $P_0(z)$ and $P_1(z)$ such that analysis filters have good frequency selectivity. In general, $P_0(z)$ can be taken as a function different from $P_1(z)$ to provide more freedom in the design. However, by choosing them to be the same, the design of the filter bank become simpler because the design of the filter bank is now reduced to that of the subfilter. Therefore, we focus on the case when $P_0(z) = P_1(z) = P(z)$. Then, the lowpass analysis filter becomes

$$H_0(z) = \frac{1}{2} \left(z^{-2N_0} + z^{-1}P(z^2) \right) \quad (5)$$

It can be verified from (5) that filter $H_0(z)$ can be an ideal lowpass filter if $P(z)$ has the following desired frequency response

$$P_d(e^{j2\omega}) = \begin{cases} e^{-j(2N_0-1)\omega}, & 0 \leq \omega \leq \omega_p \\ -e^{-j(2N_0-1)\omega}, & \omega_s \leq \omega \leq \pi \end{cases} \quad (6)$$

where ω_p , ω_s are the cutoff frequencies of the passband and stopband, respectively and $\omega_p + \omega_s = \pi$. Then, the ideal frequency response of filter $H_0(z)$ is given by

$$H_{0d}(e^{j\omega}) = \begin{cases} e^{-j2N_0\omega}, & 0 \leq \omega \leq \omega_p \\ 0, & \omega_s \leq \omega \leq \pi \end{cases} \quad (7)$$

On the other hand, with $P_0(z) = P_1(z) = P(z)$ and the ideal frequency response of $P(z)$ in (6), we can show that the highpass analysis filter,

$$H_1(z) = z^{-(2N_1+1)} - P(z^2)H_0(z), \quad (8)$$

can be an ideal highpass filter if $N_1 = 2N_0 - 1$. Then, the ideal frequency response of filter $H_1(z)$ is defined by

$$H_{1d}(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega \leq \omega_p \\ e^{-j(4N_0-1)\omega}, & \omega_s \leq \omega \leq \pi \end{cases} \quad (9)$$

In summary, the design of the lifting filter bank is reduced to finding the subfilter $P_0(z)$ such that its frequency response is the best approximate to the desired frequency response given in (6). As discussed above, two special cases where the subfilters are equiripple or maximally flat were addressed in [8]. In the following section, we will present more general method that allows to design the filter with high frequency selectivity and arbitrary regularity.

3. PROPOSED METHOD

In this section, we will formulate the design of finite impulse response (FIR) subfilter $P(z)$, which is optimally approximate to the desired frequency responses (6), as a semi-definite programming. Let us denote the transfer function of the subfilter of order N by

$$P(z) = \sum_{k=0}^N p_k z^{-k}$$

and its frequency response is a real-valued or complex-valued function of ω .

$$P(e^{j\omega}, \mathbf{p}) = \sum_{k=0}^N p_k e^{-jk\omega} = \mathbf{p}^T \cdot \mathbf{e}(\omega)$$

where $\mathbf{p} = [p_0, p_1, \dots, p_N]^T$, $\mathbf{e}(\omega) = [1, e^{-j\omega}, e^{-j2\omega}, \dots, e^{-jN\omega}]^T$.

Our goal is to find the filter coefficients $\mathbf{p} = [p_0, p_1, \dots, p_N]^T$ to minimize the maximum error between the frequency response of the filter and the desired frequency response.

That is, we solve the following minimax optimization problem minimize $\left\{ \max_{\omega \in \Omega} |P(e^{j\omega}, \mathbf{p}) - P_d(e^{j\omega})| \right\}$ for $\Omega = [0, \omega_p] \cup [\omega_s, \pi]$.

It can be cast into a constrained minimization problem.

$$\text{minimize } \eta \tag{10}$$

$$\text{subject to: } |P(e^{j\omega}, \mathbf{p}) - P_d(e^{j\omega})|^2 \leq \eta \quad \text{for } \omega \in \Omega,$$

where the filter coefficient vector \mathbf{p} is an optimization variable.

Before proceeding further, we present a brief review of semi-definite programming (SDP) [11], [14]. SDP is an optimization problem which minimize a linear or convex quadratic objective function subject to linear matrix inequality (LMI) constraints

$$\text{minimize } \mathbf{c}^T \mathbf{x}$$

$$\text{subject to: } \mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + \sum_{i=1}^n x_i \mathbf{F}_i \geq \mathbf{0}$$

where $\mathbf{x} = [x_1, \dots, x_n]^T$ is a variable vector, $\mathbf{F}_i \in \mathbf{R}^{m \times m}$ ($i = 0, \dots, n$) are given symmetric matrices, and $\mathbf{F}(\mathbf{x}) \geq \mathbf{0}$ denotes that $\mathbf{F}(\mathbf{x})$ is positive semi-definite at \mathbf{x} . It can be shown that SDP is a class of convex programming problem, and hence, its locally optimal solution is also a globally optimal one. Moreover, SDP problem can be efficiently solved by interior-point methods. There are now efficient software implementations of SDP algorithms, for example SeDuMi [12].

Our objective now is to transform the problem (10) into a semi-definite programming. First, we definite $\Omega_d = \{\omega_1, \omega_2, \dots, \omega_L\} \subset \Omega$ is a set of dense grid points in the frequency bands of interest. Then, the unconstrained optimization problem (10) can equivalently expressed as a constrained optimization problem.

$$\text{minimize } \eta \tag{11}$$

$$\text{subject to: } g_{R,p}^2(\omega_l) + g_{I,p}^2(\omega_l) \leq \eta, \quad l = 1, 2, \dots, L$$

where

$$g_{R,p}(\omega_l) = \mathbf{p}^T \text{Re}\{\mathbf{e}(\omega_l)\} + \text{Re}\{P_d(e^{j\omega_l})\}$$

$$g_{I,p}(\omega_l) = \mathbf{p}^T \text{Im}\{\mathbf{e}(\omega_l)\} + \text{Im}\{P_d(e^{j\omega_l})\}$$

Here, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ denote the real and imaginary parts of x . By using the Schur complement [11], [14], it can be shown that the constraint in (11) holds if and only if

$$\mathbf{F}_l(\mathbf{p}) = \begin{bmatrix} \eta & g_{R,p}(\omega_l) & g_{I,p}(\omega_l) \\ g_{R,p}(\omega_l) & 1 & 0 \\ g_{I,p}(\omega_l) & 0 & 1 \end{bmatrix} \geq \mathbf{0}.$$

Consequently, the optimization problem (11) can be written as minimize η (12)

subject to: $\mathbf{F}_l(\mathbf{p}) \geq \mathbf{0}$, $l = 1, 2, \dots, L$.

It can be observed that the objective is a linear function and the constraints are a set of linear matrix inequalities which can be expressed as an affine of variable \mathbf{x} , and hence the problem (12) is a semi-definite programming.

As discussed early, the regularity of filter bank is desirable in the construction of the wavelets and in certain applications. The wavelets is said to have the K -regularity if the lowpass analysis filter $H_0(z)$ and highpass filter $H_1(z)$ have K zeros at $\omega = \pi$ and $\omega = 0$, respectively. This can mathematically be expressed by

$$\left. \frac{d^k}{d\omega^k} H_0(\omega) \right|_{\omega=\pi} = \left. \frac{d^k}{d\omega^k} H_1(\omega) \right|_{\omega=0} = 0 \text{ for } k = 0, 1, \dots, K-1. \quad (13)$$

With the analysis filters given in (5) and (8), it can be verified that the regularity conditions (13) can be expressed as an linear equation

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{b} \quad (14)$$

where matrix \mathbf{A} is defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 3 & 5 & \dots & 2N+1 \\ 1 & 3^2 & 5^2 & \dots & (2N+1)^2 \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & 3^{K-1} & 5^{K-1} & \dots & (2N+1)^{K-1} \end{bmatrix}$$

and vector \mathbf{b} is given by

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2N_0 \\ (2N_0)^2 \\ \dots \\ (2N_0)^{K-1} \end{bmatrix}.$$

In summary, the filter bank design with regularity property can be formulated as a following optimization problem. minimize η (15)

subject to: $\mathbf{F}_l(\mathbf{p}) \geq \mathbf{0}$, $l = 1, 2, \dots, L$

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{b}.$$

It is important to note that the above optimization is still an semi-definite programming, and therefore, it can be efficiently solved by available softwares. In our subsequent designs, we use the popular SDP package, SeDuMi [12] to solve the problem (15).

4.SIMULATION RESULTS AND IMAGE CODING APPLICATION

In this section, we provide a design example of the lifting filter bank with regularity to illustrate the performance of our proposed method. After that, the resulting filter bank is applied to image compression, and the image coding result is compared to the Daubechies filters using in image compression standard JPEG-2000.

Our formulation (15) can be applicable to the design of nonlinear phase and linear phase FIR filters. However, in this example, we provide a design example of the linear phase filter which can be efficiently applied in image compression. The subfilter is designed with specifications: the filter order $N = 19$, edge frequencies $\omega_p = 0.4\pi$, and $\omega_s = 0.6\pi$, regularity order $K = 2$. We chose $L = 500$ samples. The magnitude responses of the analysis filters $H_0(z)$ and $H_1(z)$ are shown in Fig.2.

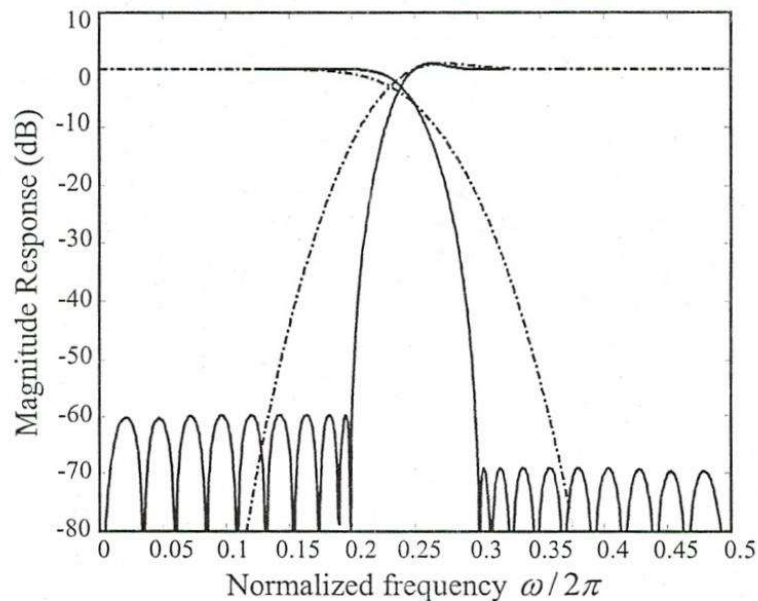


Fig.2. Magnitude responses of analysis filters designed by our method (solid line) and by Lagrange formula (dash-dotted line)

It can be seen that the filters $H_0(z)$ and $H_1(z)$ have zeros at $\omega = \pi$ and $\omega = 0$, respectively. For comparison purpose, Fig. 2 also plots the magnitude responses of maximally flat filters designed by Lagrange formula in [8]. It can be seen that by relaxing the maximally flatness condition (the maximal number of regularity order), the filters can have significantly improved frequency selectivity. Note that in addition to regularity, the frequency selectivity of filters is also desirable in many applications. It should be emphasized that our method can design the optimal filters for arbitrary regularity order while the methods in [8] can design the filters with either no regularity or the maximal number of regularity.

Furthermore, it is well-known that the scaling and wavelet functions can be generated by iterating the two-channel filter bank on its lowpass output. By applying the algorithm in [7] for 5 iterations, we obtain the analysis scaling function and wavelet function as illustrated in Fig. 3.

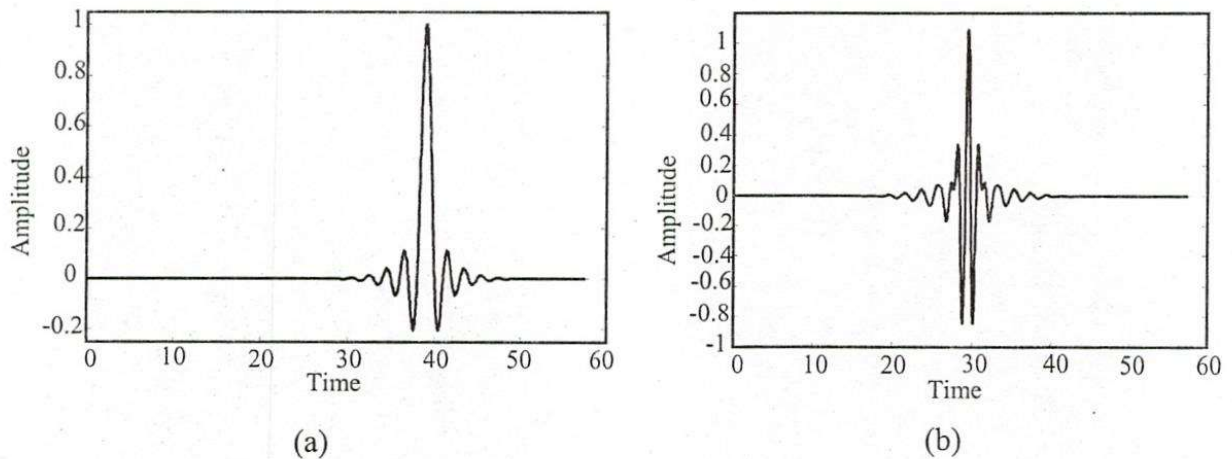


Fig. 3. (a) Analysis scaling function. (b) Analysis wavelet function

In order to evaluate the filter bank designed in image compression, we use the set partitioning in hierarchical tree codec provided in [13]. To investigate influence of the filter frequency selectivity on image coding performance, the test image used in the simulation is highly textured 8-bit image Barbara. For objective measurement of decompressed image quality, the peak signal to noise ratios (PSNR) at different bit rates are computed and plotted in Fig. 4. It can be seen in the results, our filter bank can provide improved image coding performance as compared to maximally flat Daubechies filters. For perceptual evaluation, the results of decompressed images at 1 bit per pixel (bpp) using the filter bank designed by the proposed method and using 9/7 Daubechies filters are shown in Fig. 5.

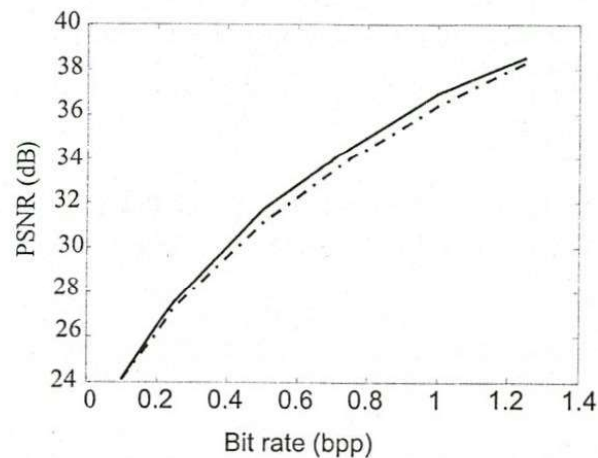
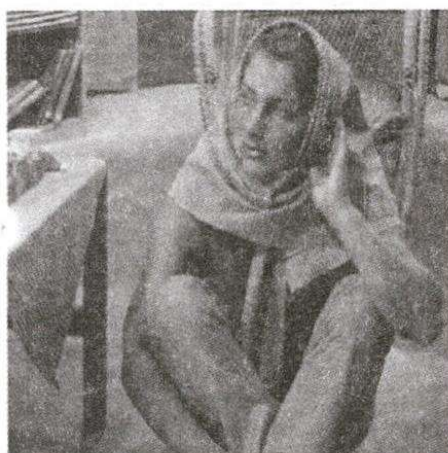


Fig. 4. PSNRs versus bit rates of the codecs using 9/7 Daubechies filters (dash-dotted line) and using our filters (solid line).



(a) PSNR: 36.4 dB



(b) PSNR: 37.0 dB

Fig. 5. Coding results at bit rate 1bpp. (a) using 9/7 Daubechies filters. (b) using filters designed by our method

5. CONCLUDING REMARKS

In this paper, the global optimization based method has been proposed to design the bi-orthogonal filter banks with arbitrary smooth order. In our method, the filter bank design problem is formulated as a semi-definite programming, so the globally optimal filter bank can be obtained. The advantage of the proposed method is that SDP problem can be flexible to incorporate the additional constraints into it, and hence, an optimal filter with regularity constraints on its frequency response can be efficiently found. Finally, the simulation results show that our filter bank can offer improved image coding performance for highly detailed images as compared to 9/7 Daubechies filters.

MỘT PHƯƠNG PHÁP TỐI ƯU CHO THIẾT KẾ DÃY BỘ LỌC WAVELETS VÀ ỨNG DỤNG TRONG NÉN ẢNH

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TÓM TẮT: Cấu trúc lifting là một công cụ hiệu quả cho việc xây dựng wavelets thế hệ thứ hai. Nó đã được dùng thực hiện các biến đổi Daubechies wavelets trong chuẩn nén ảnh JPEG-2000. Daubechies wavelets có khả năng cho chất lượng nén ảnh tốt hơn biến đổi cosine rời rạc (Discrete Cosine Transform-DCT) trong JPEG bởi vì wavelets có khả năng trình bày tín hiệu có động hơn biến đổi cosine. Tuy nhiên, trong trường hợp tỷ số nén cao, thành phần chi tiết của ảnh giải nén trong JPEG-2000 bị suy giảm, nguyên nhân là các bộ lọc Daubechies có độ phẳng tối đa (maximally flatness) trong khi đó đặc tính chọn lọc tần số rất kém. Trong bài báo này, một phương pháp thiết kế dãy bộ lọc (filter bank) và wavelets dựa trên cấu trúc lifting được trình bày. Vấn đề thiết kế dãy bộ lọc được biểu diễn như bài toán tối ưu trong đó đặc tính chọn lọc tần số của các bộ lọc được thiết kế tối ưu cho một mức độ phẳng (flatness) hoặc bậc điều hoà (regularity order) tùy ý xác định trước. Kết quả mô phỏng chỉ ra rằng dãy

bộ lọc thiết kế bằng phương pháp đề nghị có khả năng cho chất lượng ảnh giải nén tốt hơn các bộ lọc Daubechies trong JPEG-2000.

Từ khoá: Dãy bộ lọc, wavelets, nén ảnh, bậc điều hoà, chọn lọc tần số, tối ưu toàn cục.

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