EXISTENCE OF SOLUTIONS OF FUZZY CONTROL DIFFERENTIAL EQUATIONS

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ABSTRACT: *Recently, the field of differential equations has been studying in a very abstract method. Instead of considering the behaviour of one solution of a differential equation, one studies its sheaf-solution (see[10-11]). Instead of studying a differential equation, one studies differential inclusion (see[9]). Especially, one studies fuzzy differential equation (a differential equation whose variables and derivative are fuzzy sets, see[1-7]).In this paper, a fuzzy differential equation is generalized to be* ***fuzzy control differential equation (FCDE)*** *and we present the existence and comparison of solutions of (FCDE). This paper is a continuation of our works in this direction (see [10-13]).*
Keywords: Fuzzy theory; Differential equations; Control theory; Fuzzy differential equations

1. INTRODUCTION

In [1-7], the authors considered fuzzy differential equations (FDE) and had some important results on existence and comparison of solutions of FDE

$$D_H x(t) = f(t, x(t)), \qquad (1.1)$$

where

$x(t_0) = x_0 \in H_0 \subset E^n, x(t) \in E^n, t \in \left[t_0, T\right] = I \subset R_+$ and $f: I \times E^n \to E^n$.

In this paper, we consider a fuzzy control differential equation (FCDE) as following

$$D_H x(t) = f(t, x(t), u(t)), \qquad (1.2)$$

where

$x(t_0) = x_0 \in H_0 \subset E^n, x(t) \in E^n, u(t) \in E^p, t \in \left[t_0, T\right] = I \subset R_+$

and $f: I \times E^n \times E^p \to E^n$ and study existence of solutions of FCDE.

The paper is organized as follows: in section 2, we recall some basic concepts and notations which are useful in next sections. In sections 3 and 4, we present the existence of solutions and compare two solutions of FCDE.

2. PRELIMINARIES

We recall some notations and concepts presented in detail in recent series works of Lakshmikantham V. et al... (See [4-7]).

Let $K_C(R^n)$ denote the collection of all nonempty, compact and convex subsets of R^n. Given A, B in $K_C(R^n)$, the Hausdorff distance between A and B defined as

$$D[A, B] = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\right\}, \qquad (2.1)$$

where $\|.\|$ denotes the Euclidean norm in R^n.

The Hausdorff metric satisfies some below properties.

$$D\left[A+C,B+C\right]=D[A,B] \text{ and } D[A,B]=D[B,A], \tag{2.2}$$

$$D\left[\lambda A,\lambda B\right]=\lambda D[B,A], \tag{2.3}$$

$$D[A,B]\le D\left[A,C\right]+D\left[C,B\right], \tag{2.4}$$

$$D[A+A',B+B']\le D[A,B]+D\left[A',B'\right] \tag{2.5}$$

for all $A,B,C\in K_c(R^n)$ and $\lambda\in R_+$.

It is known that $(K_C(R^n), D)$ is a complete metric space and if the space $K_C(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_C(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. The fuzzy controls $u(t)$ and $\bar{u}(t)\in U\subset E^p$ were defined by definitions 1 and 5 in [10] (See p.5): for $0<\alpha\le 1$, the set $[u]^\alpha=\left\{z\in R^n : u(z)\ge\alpha\right\}$ is called the α-level set and from (i) -(iv), it follows that the α-level sets are in $K_c(R^n)$ for $0\le\alpha\le 1$.

The set $E^n=\left\{u: R^n\to[0,1]\ such\ that\ u(z) satisfies\ (i)\ to (iv)\right\}$, each it's element $u\in E^n$ is called a fuzzy set.

Let us denote

$$D_0\left[u,v\right]=sup\left\{D\left[[u]^\alpha,[v]^\alpha\right]:0\le\alpha\le 1\right\}$$

The distance between u and v in E^n, where $D\left[[u]^\alpha,[v]^\alpha\right]$ is Hausdorff distance between two sets $[u]^\alpha,[v]^\alpha$ of $K_c(R^n)$. Then, $\left(E^n,D_0\right)$ is a complete space.

Some properties of metric D_0 are similar to those of metric D above.

$$D_0\left[u+w,v+w\right]=D_0[u,v] \text{ and } D_0[u,v]=D_0[v,u], \tag{2.6}$$

$$D_0\left[\lambda u,\lambda v\right]=\lambda D_0[u,v], \tag{2.7}$$

$$D_0[u,v]\le D_0\left[u,w\right]+D_0\left[w,v\right], \tag{2.8}$$

for all $u,v,w\in E^n$ and $\lambda\in R$.

Let $u,v\in E^n$. The set $z\in E^n$ satisfying $u=v+z$ is known as the geometric difference of the sets u and $v\in E^n$ and is denoted by the symbol $u-v$. Given an interval $I=\left[t_0,T\right]\in E^n$ in R_+. We say that the mapping $F: I\to E^n$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0\in I$, if

$$\lim_{h\to 0+}\frac{F(t_0+h)-F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist in the topology of E^n and are equal to $D_H F(t_0)$. Here limits are taken in the metric space (E^n, D_0).

The Hukuhara integral of F is given by

$$\int_I F(s)ds = \left\{ \int_I f(s)ds : f \text{ is a continuous selector of } F \right\}$$

for any compact set $I \subset R_+$.

Some properties of the Hukuhara integral are in [4-7].

If $F : I \to E^n$ is integrable, one has

$$\int_{t_0}^{t_2} F(s)ds = \int_{t_0}^{t_1} F(s)ds + \int_{t_1}^{t_2} F(s)ds,\ t_0 \le t_1 \le t_2 \tag{2.9}$$

and

$$\int_{t_0}^{t} \lambda F(s)ds = \lambda \int_{t_0}^{t} F(s)ds, \lambda \in R. \tag{2.10}$$

If $F, G : I \to E^n$ are integrable, then $D_0\left[F(.), G(.)\right] : I \to R$ is integrable and

$$D_0\left[\int_{t_0}^{t} F(s)ds, \int_{t_0}^{t} G(s)ds\right] \le \int_{t_0}^{t} D_0\left[F(s), G(s)\right]ds. \tag{2.11}$$

Let us denote θ is the zero element of E^n defined as

$$\theta(z) = \begin{cases} 1 \ \text{if } z = \widehat{0}, \\ 0 \ \text{if } z \ne \widehat{0}, \end{cases}$$

Where $\widehat{0}$ is zero element of R^n.

More details in continuity, Hukuhara derivative, Hukuhara integral of the mapping $F : I \to E^n$, please see [1-7].

3. THE FUZZY DIFFERENTIAL EQUATIONS

In [1-7], authors considered the fuzzy differential equation (FDE) as following

$$D_H x(t) = f(t, x(t)),\ x(t_0) = x_0 \in E^n, \tag{3.1}$$

where $f : I \times E^n \to E^n$, state $x(t) \in E^n$.

The mapping $x \in C^1\left[I, E^n\right]$ is said to be a solution of (3.1) on I if it satisfies (3.1) on I.

Since $x(t)$ is continuous differentiable, we have

$$x(t) = x_0 + \int_{t_0}^{t} D_H x(s)ds, t \in I.$$

We associate with the initial value problem (3.1) the following

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, t \in I \tag{3.2}$$

where the integral is the Hukuhara integral. Observe that $x(t)$ is a solution of (3.1) if only it satisfies (3.2) on I.

We recall the theorems below in [1-3, 5-7].

Theorem 3.1. *Assume that*

(i) $f \in C\left[R_0, E^n\right]$, $D_0\left[f(t,x), \theta\right] \le M_0$, *on* $R_0 = I \times B(x_0, b)$ *where*

$$B(x_0, b) = \left\{x \in E^n : D_0\left[x, x_0\right] \le b\right\} \textit{ and}$$

(ii) $g \in C\left[I \times [0, 2b], \mathbb{R}_+\right]$, $0 \le g(t,w) \le M_1$ *on* $I \times [0, 2b], g(t,0) = 0$, $g(t,w)$ *is nondecreasing in w for each* $t \in I$ *and* $w(t) \equiv 0$ *is the unique solution of*

$$w' = g(t,w),\ w(t_0) = 0 \textit{ on } I. \tag{3.3}$$

(iii) $D_0\left[f(t, \overline{x}(t)), f(t,x)\right] \le g\left(t, D_0\left[\overline{x}, x\right]\right)$ *on* R_0.

Then, the (3.1) has a unique solution $x(t) = x(t, x_0)$ *on* $\left[t_0, t_0 + \eta\right]$, *where* $\eta = min\left\{a, \dfrac{b}{M}\right\}$, $M = max\left\{M_0, M_1\right\}$.

Theorem 3.2. *Assume that* $f \in C\left[\mathbb{R}_+ \times E^n, E^n\right]$ *and*

$$D_0\left[f(t,x), \theta\right] \le g(t, D_0\left[x, \theta\right]),\ (t,x) \in \mathbb{R}_+ \times E^n,$$

where $g \in C\left[\mathbb{R}_+^2, \mathbb{R}_+\right]$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$ *and the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t,w),\ w(t_0) = w_0 \ge 0$$

exists on $\left[t_0, +\infty\right)$. *Suppose further that* f *is smooth enough to guarantee local existence of solution of (3.1) for any* $(t_0, x_0) \in \mathbb{R}_+ \times E^n$. *Then the largest interval of existence of any solution* $x(t) = x(t, t_0, x_0)$ *of (3.1) such that* $D_0\left[x_0, \theta\right] \le w_0$ *is* $\left[t_0, +\infty\right)$.

4. MAIN RESULTS

In this paper, we provide a fuzzy control differential equation (FCDE) as following

$$D_H x(t) = f(t, x(t), u(t)),\ x(t_0) = x_0 \in E^n, \tag{4.1}$$

where $f : I \times E^n \times E^p \to E^n$, state $x(t) \in E^n$, control $u(t) \in E^p$.

The $u : I \to E^p$ is integrable, is called an admissible control. Let U be a set of all admissible controls. The mapping $x \in C^1\left[I, E^n\right]$ is said to be a solution of (4.1) on I if it satisfies (4.1) on I. Since $x(t)$ is continuous differentiable, we have

$$x(t) = x_0 + \int_{t_0}^{t} D_H x(s)ds, t \in I.$$

We associate with the initial value problem (4.1) the following

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds, t \in I \quad (4.2)$$

where the integral is the Hukuhara integral. Observe that $x(t)$ is a solution of (4.1) if only it satisfies (4.2) on I.

Now, based on the theorems 3.1-3.2 of FDE we have some existence results on solutions of FCDE.

Firstly, we have a unique existence of solution of FCDE as following.

Theorem 4.1. *Assume that*

(i) $f \in C\left[R_0, E^n\right]$, $D_0\left[f(t,x,u),\theta\right] \le M_0$, *on* $R_0 = I \times B(x_0,b) \times U$, *where*

$B(x_0,b) = \left\{x \in E^n : D_0\left[x,x_0\right] \le b\right\}$ *and*

(ii) $g \in C\left[I \times [0,2b], \mathbb{R}_+\right]$, $0 \le g(t,w) \le M_1$ *on* $I \times [0,2b], g(t,0) = 0$, $g(t,w)$ *is nondecreasing in* w *for each is* $t \in I$ *and* $w(t) \equiv 0$ *is unique solution of*

$$w' = g(t,w), w(t_0)=0 \text{ on } I. \quad (4.3)$$

(iii) $D_0\left[f(t,\overline{x}(t),\overline{u}(t)), f(t,x,u)\right] \le g\left(t, D_0\left[\overline{x},x\right]\right)$ *on* R_0.

Then, the (4.1) has a unique solution $x(t) = x(t,x_0,u(t))$ *on* $\left[t_0, t_0 + \eta\right]$, *where* $\eta = min\left\{a, \dfrac{b}{M}\right\}$, $M = max\left\{M_0, M_1\right\}$.

Proof. Function $u(t)$ is of variable t. Set $h(t,x(t)) = f(t,x(t),u(t))$ plays the role of function $f(t,x(t))$ in theorems 3.1 and consider $u(t)$ as parameter, then using theorems 3.1, we have theorems 4.1.

Then, we have the global existence of solution of FCDE as below.

Theorem 4.2. *Assume that* $f \in C\left[\mathbb{R}_+ \times E^n \times E^p, E^n\right]$ *and*

$$D_0\left[f(t,x,u),\theta\right] \le g(t, D_0\left[x,\theta\right]), \ (t,x,u) \in \mathbb{R}_+ \times E^n \times U,$$

where $g(t,w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$ *and the maximal solution* $r(t,t_0,w_0)$ *of*

$$w' = g(t,w), w(t_0)=w_0 \ge 0$$

exists on $\left[t_0,+\infty\right)$. *Suppose further that f is smooth enough to guarantee local existence of solution of (4.1) for any* $(t_0,x_0,u) \in \mathbb{R}_+ \times E^n \times U$. *Then the largest interval of existence of any solution* $x(t) = x(t,t_0,x_0,u(t))$ *of (4.1) such that* $D_0\left[x_0,\theta\right] \le w_0$ *is* $\left[t_0,+\infty\right)$.

Proof. Using theorem 3.2 and the proof is similar the proof of theorem 4.1.

For comparison solutions of FCDE we need the following assumption.

Assumption 4.1

The function $f : \mathbb{R}_+ \times E^n \times E^p \to E^n$ satisfies the condition

$$D_0\left[f(t,\overline{x}(t),\overline{u}(t)), f(t,x(t),u(t))\right] \leq c(t)\left\{D_0\left[\overline{x}(t),x(t)\right] + D_0\left[\overline{u}(t),u(t)\right]\right\} \quad (4.4)$$

for $\quad t \in I; \overline{x}(t), x(t) \in E^n;\ \overline{u}(t), u(t) \in E^p$,

where $c(t)$ is a positive and integralble on I.

Let $C = \int_{t_0}^{T} c(t)dt$. Because $c(t)$ is integrable on I, it is bounded almost everywhere by a positive constant K.

The below theorem indicates that solutions of FCDE depend continuously on initials and controls.

Theorem 4.2. *Suppose that f satisfies assumption 4.1 and $\overline{x}(t), x(t)$ are solutions of (4.1) starting at $\overline{x}_0, x_0$ and of the controls $\overline{u}(t), u(t)$, respectively. Then one has*

$$D_0\left[\overline{x}(t),x(t)\right] \leq \varepsilon \quad \textit{if} \quad D_0\left[\overline{u}(t),u(t)\right] \leq \delta(\varepsilon) \quad \textit{and} \quad D_0\left[\overline{x}_0, x_0\right] \leq \delta(\varepsilon) .$$

Proof.

The solutions of (4.1) for controls $\overline{u}(t), u(t)$ originating at $\overline{x}_0, x_0$, respectively, are equivalent to the following integral forms

$$\overline{x}(t) = \overline{x}_0 + \int_{t_0}^{t} f(s,\overline{x}(s),\overline{u}(s))ds$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s),u(s))ds .$$

We estimate

$$\begin{aligned}
&D_0\left[\overline{x}(t),x(t)\right] \\
&= D_0\left[\overline{x}_0 + \int_{t_0}^{t} f(s,\overline{x}(s),\overline{u}(s))ds, x_0 + \int_{t_0}^{t} f(s,x(s),u(s))ds\right] \\
&\leq D_0\left[\overline{x}_0, x_0\right] + D_0\left[\int_{t_0}^{t} f(s,\overline{x}(s),\overline{u}(s))ds, \int_{t_0}^{t} f(s,x(s),u(s))ds\right] \\
&\leq D_0\left[\overline{x}_0, x_0\right] + \int_{t_0}^{t} D_0\left[f(s,\overline{x}(s),\overline{u}(s)), f(s,x(s),u(s))\right]ds
\end{aligned}$$

$$\leq D_0\left[\overline{x}_0, x_0\right] + \int_{t_0}^{t} c(s)\left\{D_0\left[\overline{x}(s), x(s)\right] + D_0\left[\overline{u}(s), u(s)\right]\right\} ds$$

$$\leq \overline{D}\left[\overline{x}_0, x_0\right] + \int_{t_0}^{t} c(s) D_0\left[\overline{x}(s), x(s)\right] ds + \int_{t_0}^{t} c(s) D_0\left[\overline{u}(s), u(s)\right] ds .$$

Here we have used (2.4), (2.7), (2.8) and (4.4).

If $D_0\left[\overline{u}(t), u(t)\right] \leq \delta(\varepsilon)$ and $D_0\left[\overline{x}_0, x_0\right] \leq \delta(\varepsilon)$, then

$$D_0\left[\overline{x}(t), x(t)\right] \leq (K+1)\delta(\varepsilon) + \int_{t_0}^{t} c(s) D_0\left[\overline{x}(s), x(s)\right] ds .$$

Using Gronwall inequality, we have

$$D_0\left[\overline{x}(t), x(t)\right] \leq (K+1)\delta(\varepsilon) exp(C) .$$

It follows the proof if we choose $0 < \delta(\varepsilon) \leq \dfrac{\varepsilon}{(K+1) exp(C)}$.

The proof is completed.

5. CONCLUSION

In this paper we give a new concept of a fuzzy control differential equation and study its first existence results on solutions and comparison of two solutions. The fuzzy differential equation is generated from the ordinary differential equation. Also, the fuzzy control differential equation is generated from the classical control differential equation. In this paper, the control plays the role of the parameter. We need the controllableness and more character of a control. However, the study on the fuzzy differential equation and the fuzzy control differential equation is very difficult because $\left(E^n, D_0\right)$ is only complete metric space and its structure is very simple. Some more results on existence and comparison of solutions of the fuzzy control differential equation will be presented in next works [10-13].

SỰ TỒN TẠI NGHIỆM CỦA PHƯƠNG TRÌNH VI PHÂN ĐIỀU KHIỂN MỜ

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TÓM TẮT: *Gần đây, lĩnh vực phương trình vi phân đã được nghiên cứu một cách trừu tượng hơn. Thay vì khảo sát dáng điệu của một nghiệm, người ta đã khảo sát một bó nghiệm (tập các nghiệm). Thay vì nghiên cứu một phương trình vi phân, người ta nghiên cứu một bao vi phân (xem [9]). Đặc biệt, người ta đã nghiên cứu phương trình vi phân mờ là phương trình vi phân mà cả biến và đạo hàm của nó đều là các tập mờ (xem [1-7]). Trong bài báo này, chúng tôi tổng quát hoá phương trình vi phân mờ thành phương trình vi phân điều khiển mờ, trình bày sự những kết quả ban đầu về sự tồn tại nghiệm và so sánh các nghiệm của nó. Bài báo này là sự tiếp nối của các công trình của chúng tôi về hướng nghiên cứu này (xem [10-13]).*

Từ khoá: Lý thuyết mờ, Phương trình vi phân, Lý thuyết điều khiển, Phương trình vi phân mờ, Phương trình vi phân điều khiển mờ.

REFERENCES

[1]. Wu. C, Song. S., *Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations,* Journal of Mathematical Analysis and Applications, 202, 629-644, (1996)

[2].Kaleva. O., *Fuzzy differential equations,* Fuzzy Sets and Systems, 24, 301-317, (1987).

[3].Kaleva. O., *The Cauchy problem for fuzzy differential equations,* Fuzzy Sets and Systems, 35, 389-396, (1990).

[4].Lakshmikantham. V., *Set differential equations versus fuzzy differential equations,* Applied Mathematics and Computation 164 277-294, (2005).

[5].Lakshmikantham V, Gnana Bhaskar T, Vasundhara Devi J., *Theory of set differential equations in metric spaces,* Cambridge Scientific Publisher, UK, (2006).

[6].Lakshmikantham V, Mohapatra R., *Theory of fuzzy differential equations and inclusions,* Taylor & Francis, London, (2003).

[7].LakshmikanthamV., Leela S., *Fuzzy differential systems and the new concept of stability,* Nonlinear Dynamics and Systems Theory, 1(2), 111-119, (2001).

[8].Phu N. D, *Genaral views in theory of systems,* VNU Publishing House, HCM City, (2003).

[9].Phu N.D., Huong N.T., *Multivalued Differential Equations,*VNU Publishing House, HCM City, (2005).

[10].Phu N. D., Tung T.T., *Sheaf optimal control problems in fuzzy type,* J. Science and Technology Development 8 (12), 5-11, (2005).

[11].Phu N. D., Tung T.T., *The comparison of sheaf- solutions in fuzzy control problems,* J. Science and Technology Development 9 (2), 5-10, (2006).

[12].Phu N. D., Tung T.T., *Some Results on Sheaf solutions of Sheaf fuzzy Control Problems,* Electronic Journal of Differential Equations, Vol N. 108, pp 1-8, (2006).

[13].Phu N. D., Tung T.T., *Some Properties of Sheaf solutions of Sheaf set Control Problems,* J. Nonlinear Analysis, Vol 67, pp 1309-1315, (2007).

[14].Tolstonogov A., *Differential Inclusions in a Banach Space,* Kluwer Academic Publishers, Dordrecht, (2000).