APPROXIMATE OPTIMALITY CONDITIONS AND DUALITY FOR CONVEX INFINITE PROGRAMMING PROBLEMS

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ABSTRACT: *Necessary and sufficient conditions for ε-optimal solutions of convex infinite programming problems are established. These Kuhn-Tucker type conditions are derived based on a new version of Farkas' lemma proposed recently. Conditions for ε-duality and ε-saddle points are also given.*
Keywords: ε-solution, ε-duality, ε-saddle point.

1. INTRODUCTION

The study of approximate solutions of optimization problems has been received attentions of many authors (see [6], [7], [9], [10], [11], [12] and references therein). Many of these papers deal with convex problems in finite/infinite dimensional spaces and finite number of convex inequality constraints and affine equality constraints. The others deal with Lipschitz problems or vector optimization problems. In order to establish approximate optimality conditions the authors often used Slater type constraint qualification (see, e.g., [7], [11], and [12]). Recently, Scovel, Hush and Steinwart [13] introduced a general treatment of approximate duality theory for convex programming problems (with a finite number of constraints) on a locally convex Hausdorff topological vector space.

In the recent years, convex problems in infinite dimensional setting with possibly infinite number of constraints were studied in [2], [3], where the optimality conditions, duality results, and saddle-point theorems were established, based on the conjugate theory in convex analysis and a new closedness condition called (CC) instead of Slater condition.

In this paper, we consider a model of convex infinite programming problem, that is, a convex problem in infinite dimensional spaces with infinitely many inequality constraints. We study the necessary and sufficient conditions for a feasible point to be an ε-solution, approximate duality and approximate saddle-points, using the tools introduced in [2] and [3]. These results will be established based upon a new Farkas type result in [3] and under the closedness condition (CC).

The paper is organized as follows: Section 2 is devoted to some basic definitions and basic lemmas which will be used later on. In Section 3, several ε-optimality conditions of Karush-Kuhn-Tucker type for an approximate solution of a class of convex infinite programming problems are established. In particular, an optimality condition for (exact) solution of these problems are derived as a consequence of the corresponding approximate result. Finally, results on approximate duality and on approximate saddle-points are established in the last section, Section 4. An example is given to illustrate the significance of the results.

2. PRELIMINARIES

Let T be an arbitrary (possibly infinite) index set and let R^T be the product space with product topology. Denote by $R^{(T)}$ the space of all generalized sequences $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in R$ for each $t \in T$ and the set $\operatorname{supp}\lambda := \{ t \in T \mid \lambda_t \neq 0 \}$, the supporting set of λ, is a finite subset of T. Set

$$R_+^{(T)} := \left\{ \lambda = (\lambda_t) \in R^{(T)} \mid \lambda_t \geq 0, t \in T \right\}.$$

Note that $R_+^{(T)}$ is a convex cone in $R^{(T)}$ (see [5], page 48).

We recall some notations and basic results which will be used later on. Let X be a locally convex Hausdorff topological vector space with its topological dual, X^*, endowed with weak*-topology. For a subset $D \subset X$, the closure of D and the convex cone generated by D are denoted by $\operatorname{cl} D$ and $\operatorname{cone} D$, respectively.

Let $f : X \to R \cup \{+\infty\}$ be a proper lower semi-continuous (l.s.c.) and convex function. The conjugate function of f, f^*, is defined as

$$f^* : X^* \to R \cup \{+\infty\},$$
$$f^*(v) := \sup \{ v(x) - f(x) \mid x \in \operatorname{dom} f \},$$

where $\operatorname{dom} f := \{ x \in X \mid f(x) < +\infty \}$ is the effective domain of *f*. The epigraph of *f* is defined by

$$\operatorname{epi} f := \{ (x, r) \in X \times R \mid f(x) \leq r \}.$$

The subdifferential of the convex function *f* at $a \in \operatorname{dom} f$ is the set (possibly empty)

$$\partial f(a) := \left\{ v \in X^* \mid f(x) - f(a) \geq v(x-a), \forall x \in X \right\}.$$

For $\varepsilon \geq 0$, the ε-subdifferential of *f* at $a \in \operatorname{dom} f$ is defined as the set (possibly empty)

$$\partial_\varepsilon f(a) := \left\{ v \in X^* \mid f(x) - f(a) \geq v(x-a) - \varepsilon, \forall x \in \operatorname{dom} f \right\}.$$

If $\varepsilon > 0$ then $\partial_\varepsilon f(a)$ is nonempty and it is a weak*-closed subset of X^*. When $\varepsilon = 0$, $\partial_0 f(a)$ collapses to $\partial f(a)$.

For any $a \in \operatorname{dom} f$, $\operatorname{epi} f^*$ has a representation as follows (see [8]):

$$\operatorname{epi} f^* = \bigcup_{\varepsilon \geq 0} \{ (v, v(a) + \varepsilon - f(a)) \mid v \in \partial_\varepsilon f(a) \} \tag{2.1}$$

Noting that, for $\varepsilon_1, \varepsilon_2 \geq 0$ and $z \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\partial_{\varepsilon_1} f(z) + \partial_{\varepsilon_2} g(z) \subset \partial_{\varepsilon_1 + \varepsilon_2} (f+g)(z)$$

and for $\mu > 0, \varepsilon \geq 0, z \in \operatorname{dom} f$ (see [14], page 83),

$$\mu \partial_\varepsilon f(z) = \partial_{\mu\varepsilon} (\mu f)(z), \tag{2.2}$$

Let us denote by $\delta_B(x)$ the indicator function of a subset *B* of *X*, i.e.,

$$\delta_B(x) := \begin{cases} 0, & x \in B, \\ +\infty, & x \notin B. \end{cases}$$

Let C be a closed convex subset of X. For $\varepsilon \geq 0$, the ε-normal cone of C at z, denoted by $N_\varepsilon(C,z)$, is defined by

$$N_\varepsilon(C,z) := \left\{ u \in X^* \mid u(x-z) \leq \varepsilon, \ \forall x \in C \right\}.$$

It is easy to see that $N_\varepsilon(C,z) = \partial_\varepsilon \delta_C(z)$. Let $f_t : X \to R \bigcup \{+\infty\}, t \in T$, be proper, l.s.c. and convex functions. We shall deal with the following convex system:

$$\sigma := \{ f_t(x) \leq 0, \ \forall t \in T, \ x \in C \}.$$

Denote by A the solution set of σ, that is, $A := \left\{ x \in X \mid x \in C, f_t(x) \leq 0, \forall t \in T \right\}$. The system σ is said to be consistent if $A \neq \phi$. The cone

$$K := \operatorname{cone}\left\{ \bigcup_{t\in T} \operatorname{epi} f_t^* \bigcup \operatorname{epi} \delta_C^* \right\}$$

is called the characteristic cone of σ. A consistent system σ is said to be a **Farkas-Minkowski system** (FM) if K is weak*-closed. The (FM) condition was introduced recently in [2]. It was known that (FM) condition is weaker than several known interior- type constraint qualifications. The following **closedness condition** [2] will be used later on.

$$\text{(CC)}: \quad \operatorname{epi} f^* + \operatorname{cl} K \textit{ is } \text{weak}^* - \text{closed}.$$

Remark 2.1 *If* σ *is* (FM) *and f is continuous at least one point in C then the condition* (CC) *is satisfied (see Theorem 1 in [3]; see also [1, 2]).*

The following lemma will be used as a main tool to establish -optimality conditions and related results for convex infinite problems. It is known as generalized Farkas' lemma and was established recently in [3].

Lemma 2.1 *[3] Suppose that* σ *is* (FM) *and* (CC) *holds. For any* $\alpha \in R$, *the following statements are equivalent:*

(i) $x \in C, f_t(x) \leq 0, \forall t \in T \ \Rightarrow \ f(x) \geq \alpha$;

(ii) $(0,-\alpha) \in \operatorname{epi} f^* + K$;

(iii) $\exists \lambda \in R_+^{(T)} : f(x) + \sum_{t\in T} \lambda_t f_t(x) \geq \alpha, \forall x \in C.$

3. APPROXIMATE OPTIMALITY CONDITIONS

Consider the following optimization problem:

(P) Minimize $f(x)$

subject to $f_t(x) \leq 0, \forall t \in T,$

$x \in C,$

where T is an arbitrary (possibly infinite) index set, X is a locally convex Hausdorff topological vector space, $f, f_t : X \to R \bigcup \{+\infty\}, t \in T$, are proper, l.s.c and convex functions, C is a closed convex subset of X. Denote by A the feasible set of (P), i.e.,

$$A = \left\{ x \in X \mid x \in C, f_t(x) \leq 0, \ \forall t \in T \right\}.$$

From now on, assume that $A \neq \phi$ and inf(P) is finite. The definition of ε-solution for a convex problem with finite number of constraints was presented in [12]. We present the definition of ε-solution for convex infinite problem (P) as follows.

Definition 3.1 For the problem (P), let $\varepsilon \geq 0$. A point $z \in A \cap \operatorname{dom} f$ is said to be an ε-solution of (P) if $f(z) \leq \inf(P) + \varepsilon$, i.e., $f(z) \leq f(x) + \varepsilon$ for all $x \in A$.

It is worth noting that a point $z \in A$ is an ε-solution of (P) if and only if $0 \in \partial_\varepsilon (f + \delta_A)(z)$. We now give a characterization of ε-optimality condition for (P).

Theorem 3.1 *Let $\varepsilon \geq 0$ and let $z \in A \cap dom f$. Suppose that σ is* (FM) *and that* (CC) *holds. Then z is an -solution of* (P) *if and only if there exist $\lambda = (\lambda_t) \in R_+^{(T)}$, $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ and $\varepsilon_t \geq 0$ for all $t \in T$, such that*

$$0 \in \partial_{\varepsilon_1} f(z) + \sum_{t \in \operatorname{supp} \lambda} \partial_{\varepsilon_t} (\lambda_t f_t)(z) + N_{\varepsilon_2}(C, z), \tag{3.1}$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \sum_{t \in \operatorname{supp} \lambda} \varepsilon_t - \sum_{t \in \operatorname{supp} \lambda} \lambda_t f_t(z). \tag{3.2}$$

Proof. Suppose that z is an ε-solution of (P). This means that

$$x \in C,\ f_t(x) \leq 0,\ \forall t \in T \quad \Rightarrow \quad f(x) \geq f(z) - \varepsilon. \tag{3.3}$$

Since σ is (FM) and (CC) holds, it follows from Lemma 2.1 that (3.3) is equivalent to

$$(0, \varepsilon - f(z)) \in \operatorname{epi} f^* + \operatorname{cone}(\bigcup_{t \in T} \operatorname{epi} f_t^* \bigcup \operatorname{epi} \delta_C^*).$$

Hence, there exists $\lambda = (\lambda_t) \in R_+^{(T)}$ such that

$$(0, \varepsilon - f(z)) \in \operatorname{epi} f^* + \sum_{t \in T} \lambda_t \operatorname{epi} f_t^* + \operatorname{epi} \delta_C^*.$$

From this and (2.1) (applies to $\operatorname{epi} f^*$, $\operatorname{epi} f_t^*$ and $\operatorname{epi} \delta_C^*$), there exist $u, v, u_t \in X^*$, $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$, $\varepsilon_t^{'} \geq 0$ and $u \in \partial_{\varepsilon_1} f(z)$, $u_t \in \partial_{\varepsilon_t^{'}} f_t(z)$, $v \in \partial_{\varepsilon_2} \delta_C(z)$ for all $t \in T$ such that

$$\begin{cases} 0 = u + \sum_{t \in \operatorname{supp} \lambda} \lambda_t u_t + v, \\ \varepsilon - f(z) = u(z) + \varepsilon_1 - f(z) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t [u_t(z) + \varepsilon_t^{'} - f_t(z)] + v(z) + \varepsilon_2 - \delta_C(z). \end{cases}$$

The first equality gives $0 \in \partial_{\varepsilon_1} f(z) + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \partial_{\varepsilon_t^{'}} f_t(z) + N_{\varepsilon_2}(C, z)$

and the second implies $\varepsilon = \varepsilon_1 + \varepsilon_2 + \sum_{t \in \operatorname{supp} \lambda} \lambda_t \varepsilon_t^{'} - \sum_{t \in \operatorname{supp} \lambda} \lambda_t f_t(z)$.

Let $\varepsilon_t := \lambda_t \varepsilon_t^{'}$. Taking (2.2) into account, we get

$$0 \in \partial_{\varepsilon_1} f(z) + \sum_{t \in \operatorname{supp} \lambda} \partial_{\varepsilon_t} (\lambda_t f_t)(z) + N_{\varepsilon_2}(C, z),$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \sum_{t\in \operatorname{supp}\lambda} \varepsilon_t - \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(z).$$

The necessity has been proved.

Conversely, suppose that there exist $\lambda = (\lambda_t) \in R_+^{(T)}$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ and $\varepsilon_t \geq 0$ for all $t \in T$ satisfying (3.1) and (3.2). Then there exists

$u \in \partial_{\varepsilon_1} f(z) + \sum_{t\in \operatorname{supp}\lambda} \partial_{\varepsilon_t}(\lambda_t f_t)(z)$ such that $-u \in N_{\varepsilon_2}(C,z)$.

Note that

$-u \in N_{\varepsilon_2}(C,z) \quad \Leftrightarrow \quad u(x) \geq u(z) - \varepsilon_2, \forall x \in C.$

As $u \in \partial_{\varepsilon_1} f(z) + \sum_{t\in \operatorname{supp}\lambda} \partial_{\varepsilon_t}(\lambda_t f_t)(z)$, there exist v, $u_t \in X^*$ for all $t \in \operatorname{supp}\lambda$ such that

$$u = v + \sum_{t\in \operatorname{supp}\lambda} u_t,\ v \in \partial_{\varepsilon_1} f(z),\ u_t \in \partial_{\varepsilon_t}(\lambda_t f_t)(z),\ \forall t \in \operatorname{supp}\lambda.$$

Hence, for all $x \in X$, $f(x) - f(z) \geq v(x-z) - \varepsilon_1$, and

$$\lambda_t f_t(x) - \lambda_t f_t(z) \geq u_t(x-z) - \varepsilon_t,\ \forall t \in \operatorname{supp}\lambda.$$

Thus,

$$f(x) + \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(x) - f(z) - \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(z) \geq v(x-z) + \sum_{t\in \operatorname{supp}\lambda} u_t(x-z) - (\varepsilon_1 + \sum_{t\in \operatorname{supp}\lambda} \varepsilon_t), \forall x \in X.$$

Since $u = v + \sum_{t\in \operatorname{supp}\lambda} u_t$ and $u(x-z) \geq -\varepsilon_2$ for all $x \in C$,

$$f(x) + \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(x) - f(z) - \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(z) \geq -(\varepsilon_1 + \varepsilon_2 + \sum_{t\in \operatorname{supp}\lambda} \varepsilon_t),\ \forall x \in C.$$

Combining this and (3.2) we get

$$f(x) + \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(x) \geq f(z) - \varepsilon,\ \forall x \in C.$$

Since $\lambda_t \geq 0$ and $f_t(x) \leq 0$ for all $x \in A$ and for all $t \in T$, $f(x) \geq f(z) - \varepsilon$ for all $x \in A$, which proves z to be an ε-solution of *(*P*)*.

We get the following result proved recently in [3] when taking $\varepsilon = 0$.

Corollary 3.1 *For the problem* (P), *let* $z \in A \cap \operatorname{dom} f$. *Suppose that* σ *is* (FM) *and* (CC) *holds. Then* z *is a solution of* (P) *if and only if there exists* $\lambda \in R_+^{(T)}$ *such that*

$$0 \in \partial f(z) + \sum_{t\in T} \lambda_t \partial f_t(z) + N_C(z),\ \lambda_t f_t(z) = 0, \forall t \in T.$$

Proof. Let $\varepsilon = 0$. It follows from (3.2) that $0 = \varepsilon_1 + \varepsilon_2 + \sum_{t\in \operatorname{supp}\lambda} \varepsilon_t - \sum_{t\in \operatorname{supp}\lambda} \lambda_t f_t(z)$.

The conclusion follows by taking the fact that $\lambda_t f_t(z) \leq 0$ for each $t \in T$, $\varepsilon_1, \varepsilon_2 \geq 0$ and $\varepsilon_t \geq 0$ for all $t \in T$ into account.

Corollary 3.2 *Let $\varepsilon \geq 0$ and let $z \in A \cap dom\, f$. For the Problem* (P), *assume that $f, f_t, t \in T$, are finite-valued, continuous, and convex functions. Assume further that the system σ is* (FM). *Then z is an ε-solution of* (P) *if and only if there exist $\lambda = (\lambda_t) \in R_+^{(T)}$, $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ and $\varepsilon_t \geq 0$ for all $t \in T$ such that*

$$0 \in \partial_{\varepsilon_1} f(z) + \sum_{t \in \operatorname{supp} \lambda} \partial_{\varepsilon_t} (\lambda_t f_t)(z) + N_{\varepsilon_2}(C, z),$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \sum_{t \in \operatorname{supp} \lambda} \varepsilon_t - \sum_{t \in \operatorname{supp} \lambda} \lambda_t f_t(z).$$

Proof. The conclusion follows from Remark 2.1 and Theorem 3.1.

Example

Consider the problem

(Q) *Minimize* x^2

subject to $tx^2 - x \leq 0,\ t \in [0,1],$

$x \in C = [-1/2, 1/2].$

The feasible set of (Q) is $A = [0, 1/2]$ and so $\alpha = \inf(\mathrm{Q}) = 0$. To illustrate Theorem 3.1, take $\varepsilon = 1/4$ and $z = 1/2$. We will show that there exist $\lambda \in R_+^{(T)}, \varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ and $\varepsilon_t \geq 0$ for all $t \in T$ such that (3.1) and (3.2) hold.

Set $f(x) = x^2,\ f_t(x) = tx^2 - x,\ t \in T = [0,1]$. A simple computation gives

$$\partial_{\varepsilon_1} f(1/2) = \left\{ u \,\middle|\, 1 - 2\sqrt{\varepsilon_1} \leq u \leq 1 + 2\sqrt{\varepsilon_1} \right\} \text{ and } N_{\varepsilon_2}(C, 1/2) = \{ v \mid v \geq -\varepsilon_2 \}.$$

If we choose

$$\varepsilon_1 = \varepsilon_2 = 1/8,\ u = 1/8 \in \partial_{\varepsilon_1} f(1/2),\ v = -1/8 \in N_{\varepsilon_2}(C, 1/2)$$

then

$$0 = u + v \in \partial_{\varepsilon_1} f(1/2) + N_{\varepsilon_2}(C, 1/2).$$

Letting $\lambda = (\lambda_t) = (0_t)$ and $\varepsilon_t = 0$ for all $t \in T$, we obtain

$$0 \in \partial_{\varepsilon_1} f(1/2) + \sum_{t \in T} \lambda_t \partial_{\varepsilon_t} f(1/2) + N_{\varepsilon_2}(C, 1/2)$$

and

$$1/4 = \varepsilon = \varepsilon_1 + \varepsilon_2 + \sum_{t \in T} \lambda_t \varepsilon_t - \sum_{t \in T} \lambda_t f_t(1/2).$$

Thus, (3.1) and (3.2) are satisfied and $z = 1/2$ is an $(1/4)$-solution of (Q).

4. ε-DUALITY AND ε-SADDLE POINT

The study of ε-duality and ε-saddle points of an optimization problem was seen in many papers (see [4], [9], [10], [11], [12], [13]). There, the problems in consideration have a finite number of constraints. In this section we establish some results concerning ε-duality and ε-saddle points of the convex infinite problem (P) introduced in Section 3. For the problem (P), the Lagrangian function (see [2]) is

$$L(x,\lambda)=\begin{cases} f(x)+\sum_{t\in T}\lambda_t f_t(x), x\in C, \lambda\in R_+^{(T}, \\ +\infty, \quad \text{otherwise.}\end{cases}$$

Set $\psi(\lambda)=\inf_{x\in C} L(x,\lambda)$, $\lambda\in R_+^{(T)}$. The following optimization problem is called the Lagrange dual problem of (P) [2]:

$$\text{(D)} \qquad \sup \psi(\lambda)$$
$$\text{subject to } \lambda\in R_+^{(T)}.$$

Definition 4.1 For the problem (D), let $\varepsilon\geq 0$ and let $\bar{\lambda}$ be a point of $R_+^{(T)}$. The point $\bar{\lambda}$ is said to be an ε-solution of (D) if $\psi(\bar{\lambda})\geq \sup(\text{D})-\varepsilon$, i.e., $\psi(\bar{\lambda})\geq\psi(\lambda)-\varepsilon$ for all $\lambda\in R_+^{(T)}$.

Theorem 4.1 Let $\varepsilon\geq 0$. *Suppose that* σ *is* (FM) *and* (CC) *holds. If* z *is an* ε*-solution of* (P) *then there exists* $\bar{\lambda}\in R_+^{(T)}$ *such that* $\bar{\lambda}$ *is an* ε*-solution of* (D).

Proof. Denote by S_ε and D_ε the sets of all ε-solutions of (P) and (D), respectively. Since $S_\varepsilon\subset A\subset C$, $\psi(\lambda)=\inf_{x\in C} L(x,\lambda)\leq\inf_{x\in A} L(x,\lambda)\leq\inf_{x\in S_\varepsilon} L(x,\lambda)$.

Hence,

$$\psi(\lambda)\leq L(x,\lambda)\leq f(x), \forall x\in S_\varepsilon, \forall\lambda\in R_+^{(T)}.$$

Since z is an ε-solution of (P),

$$\psi(\lambda)\leq f(z),\ \forall\lambda\in R_+^{(T)}. \tag{4.1}$$

On the other hand, if z is an ε-solution of (P) then

$$f_t(x)\leq 0, \forall t\in T, x\in C \quad\Rightarrow\quad f(x)\geq f(z)-\varepsilon.$$

Since σ is (FM) and (CC) holds, by Lemma 2.1, there exists $\bar{\lambda}\in R_+^{(T)}$ such that

$$f(z)-\varepsilon\leq f(x)+\sum_{t\in T}\bar{\lambda}_t f_t(x),\ \forall x\in C.$$

Hence, $f(z)-\varepsilon\leq\psi(\bar{\lambda})$. This and (4.1) imply that $\psi(\lambda)-\varepsilon\leq\psi(\bar{\lambda})$ for all $\lambda\in R_+^{(T)}$. Thus, $\bar{\lambda}$ is an ε-solution of (D).

Remark 4.1 Let $\varepsilon\geq 0$ and let $z\in A\cap\operatorname{dom} f$. If there exists $\lambda\in R_+^{(T)}$ such that $f(z)-\varepsilon\leq\psi(\lambda)$ then it is easy to see that z is an ε-solution of (P).

We now give a definition of ε-saddle points of (P).

Definition 4.2 Let $\varepsilon\geq 0$. A point $(z,\bar{\lambda})\in C\times R_+^{(T)}$ is said to be an ε-saddle point of the Lagrange function L if $L(z,\lambda)-\varepsilon\leq L(z,\bar{\lambda})\leq L(x,\bar{\lambda})+\varepsilon$ for any $(x,\lambda)\in C\times R_+^{(T)}$.

Theorem 4.3 *Suppose that* σ *is* (FM) *and* (CC) *holds. Let* $\varepsilon \geq 0$ *and let* $z \in A \cap \operatorname{dom} f$. *If* z *is an* ε*-solution of* (P) *then there exists* $\overline{\lambda} \in R_{+}^{(T)}$ *such that* $(z, \overline{\lambda})$ *is an* ε*-saddle point of the Lagrange function L.*

Proof. Suppose that $z \in A \cap \operatorname{dom} f$ is an ε-solution of *(P)*. Then

$$x \in C, f_t(x) \leq 0, \forall t \in T \quad \Rightarrow \quad f(x) \geq f(z) - \varepsilon.$$

Since σ is (FM) and (CC) holds, it follows from Lemma 2.1 that there exists $\overline{\lambda} \in R_{+}^{(T)}$ satisfying

$$f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq f(z) - \varepsilon, \ \forall x \in C. \tag{4.2}$$

An argument as in the proof of Theorem 4.1 shows that $\overline{\lambda}$ is also an ε-solution of (D). Since $z \in A$, we get $f_t(x) \leq 0$ for all $t \in T$. Hence,

$$f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) + \varepsilon \geq f(z) \geq f(z) + \sum_{t \in T} \overline{\lambda}_t f_t(z), \ \forall x \in C,$$

or, equivalently, $L(x, \overline{\lambda}) + \varepsilon \geq L(z, \overline{\lambda})$ for all $x \in C$. On the other hand, since $z \in S_{\varepsilon}$, $f_t(z) \leq 0$ for all $t \in T$. Then,

$$L(z, \lambda) = f(z) + \sum_{t \in T} \lambda_t f_t(z) \leq f(z), \ \forall \lambda \in R_{+}^{(T)}. \tag{4.3}$$

Moreover, it follows from (4.2) that, $f(z) \leq L(z, \overline{\lambda}) + \varepsilon$. This, together with (4.3), implies that $L(z, \lambda) - \varepsilon \leq L(z, \overline{\lambda})$ for all $\lambda \in R_{+}^{(T)}$. Consequently, for all $x \in C$ and for all $\lambda \in R_{+}^{(T)}$, $L(z, \lambda) - \varepsilon \leq L(z, \overline{\lambda}) \leq L(x, \overline{\lambda}) + \varepsilon$.

Theorem 4.4 *Let* $\varepsilon \geq 0$. *If* $(z, \overline{\lambda})$ *is an* $(\varepsilon/2)$*-saddle point of the Lagrange function L then* z *is an* ε*-solution of* (P) *and* $\overline{\lambda}$ *is an* ε*-solution of* (D).**Proof.** Since $(z, \overline{\lambda}) \in C \times R_{+}^{(T)}$ is an $(\varepsilon/2)$-saddle point of the Lagrange function *L*, we have

$$f(z) + \sum_{t \in T} \lambda_t f_t(z) - (\varepsilon/2) \leq f(z) + \sum_{t \in T} \overline{\lambda}_t f_t(z) \leq f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) + (\varepsilon/2), \ \forall (x, \lambda) \in C \times R_{+}^{(T}.$$

Hence,

$$f(z) + \sum_{t \in T} \lambda_t f_t(z) \leq f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) + \varepsilon, \ \forall (x, \lambda) \in C \times R_{+}^{(T}. \tag{4.4}$$

If $x \in A$ then $f_t(x) \leq 0$ for all $t \in T$, and hence, $\sum_{t \in T} \overline{\lambda}_t f_t(x) \leq 0$.

Taking $\lambda = 0$ and noting that $f(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \leq f(x)$ for all $x \in A$, it follows from

(4.4)

that $f(z) \leq f(x) + \varepsilon$ for all $x \in A$, i.e., z is an ε-solution of (P). Since $z \in C$,

$$\inf_{x \in C} \{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \} \leq f(z) + \sum_{t \in T} \lambda_t f_t(z).$$

It follows from (4.4) that

$$\inf_{x\in C}\{f(x)+\sum_{t\in T}\lambda_t f_t(x)\} \le f(z)+\sum_{t\in T}\lambda_t f_t(z) \le \inf_{x\in C}\{f(x)+\sum_{t\in T}\overline{\lambda}_t f_t(x)\}+\varepsilon.$$

Hence, $\psi(\lambda)-\varepsilon \le \psi(\overline{\lambda})$, i.e., $\overline{\lambda}$ is an ε-solution of (D).

ĐIỀU KIỆN XẤP XỈ TỐI ƯU VÀ ĐỐI NGẪU CHO BÀI TOÁN QUI HOẠCH LỒI VÔ HẠN

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TÓM TẮT: *Bài báo này thiết lập các điều kiện cần và đủ tối ưu cho nghiệm xấp xỉ của bài toán qui hoạch lồi vô hạn. Các điều kiện này thuộc dạng Kuhn-Tucker và nhận được bằng cách sử dụng một kết quả dạng Farkas được thiết lập gần đây. Một số kết quả về đối ngẫu Lagrange xấp xỉ và điểm yên ngựa xấp xỉ cho bài toán lồi vô hạn cũng được thiết lập.*

Từ khoá: *ε-nghiệm, ε-đối ngẫu, điểm ε-yên ngựa.*

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