THE COMPARISON OF SHEAF- SOLUTIONS IN FUZZY CONTROL PROBLEM

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ABSTRACT: *In [2] the author considered the Sheaf-Optimal Control Problem (SOCP) by differential equations:* $\frac{dx(t)}{dt} = f(t, x(t), u(t))$,

where $x_0 \in Q \subset R^n, u \in U \subset R^p, t \in [0,T] \subset R^+$, *and sheaf of solutions:*

$$H_{t,u} = \{x(t) = x(t, x_0, u(t)) \mid x_0 \in H_0 \subseteq Q, t \in I = [0,T] \subset R^+, u(t) \in U\}$$

with the goal function $I(u) \to \min$.

In [5], we have offered the neccesary conditions of Sheaf-Optimal Control Problem in Fyzzy type (SOFCP), that means the controls $u(t) \in U \subset E^p$ *not belong to* R^p .

This paper shows some comparison of sheaf-solutions $H_{t,u}$ *and* $H_{t,\bar{u}}$ *for many kinds of fuzzy controls* $u(t), \bar{u}(t) \in U \subset E^p$ *in Sheaf Fuzzy Control Problem(SFCP)*

Keywords: *Fuzzy Theory, Optimal Control Theorey, Differential Equations.*

1. INTRODUCTION :

For Sheaf-Optimal Control Problem (SOCP) many controls $u(t)$ and $\bar{u}(t) = u(t) + \Delta u$ are considered with $\|\Delta u\| = \|\bar{u}(t) - u(t)\| \le \delta$, where $u(t), \bar{u}(t) \in U \subset R^p$ [2]. For Sheaf-Optimal Control Problem in Fuzzy Type (SOFCP) we have fuzzy controls $u(t)$ and $\bar{u}(t) \in U \subset E^p$ with $\|\bar{u}(t) - u(t)\| \le T\sqrt{p}$ [5].

For the Sheaf Fuzzy Control Problem (SFCP) we have the same fuzzy controls $u(t)$ and $\bar{u}(t) \in U \subset E^p$, that was defined by definition 5 in [5]. The paper is organized as follows:

In the second section, offering the Sheaf Fuzzy Control Problem (SFCP) we get estimations of the norms $\| \bullet \|_C$ and $\| \bullet \|_L$ of

$$\Delta x = x(t, x_0, \bar{u}(t)) - x(t, x_0, u(t)) \text{ and}$$
$$\Delta f = f(t, x(t, x_0, \bar{u}(t)), \bar{u}(t)) - f(t, x(t, x_0, u(t)), u(t))$$

In section 3, we study some comparisons of sheaf solutions $H_{t,u}$ in many kinds of fuzzy controls $u(t), \bar{u}(t) \in U \subset E^p$, that means we have to compare the measure $|\mu(H_{T,\bar{u}}) - \mu(H_{T,u})|$

2. THE SHEAF FUZZY CONTROL PROBLEM (SFCP)

As we know, the solutions of differential equations depend locally on initial, right hand side and parameters. Now, we consider a control system of differential equations

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)) \qquad (1)$$

and $f: I \times R^n \times E^p \to R^n$.

Definition 1. The sheaf - solution (or sheaf-trajectory) $\rfloor x(t, x_0, u)\lceil$ which gives at the time t a set

$$H_{t,u} = \{x(t) = x(t, x_0, u) | x_0 \in H_0 \subset Q, x(t) - \text{solution of}(1)\}, \quad (2)$$

where $x_0 \in H_0 \subset Q \subset R^n$, $u(t) \in U \subset E^p$, $t \in I$.

In the case, when a control u(t) is fuzzy, we have Sheaf Fuzzy Control Problem (SFCP).

Suppose at time $t = 0, u(0) = 0$ and $x(0) = x_0 \in H_0$. For two admissible controls u(t) and $\overline{u}(t) \in U \subset E^p$, we have two sets of sheaf-solutions

$$H_{t,u} = \{x(t) = x(t, x_0, u) | \; x_0 \in H_0 \subset Q, x(t) - \text{a solution of}(1) \text{ by control } u(t)\}$$

$$H_{t,\overline{u}} = \{\overline{x}(t) = x(t, x_0, \overline{u}(t)) | x_0 \in H_0 \subset Q, \overline{x}(t) - \text{a solution of}(1) \text{ by control } \overline{u}(t)\},$$

where $t \in I$. (See fig.1)

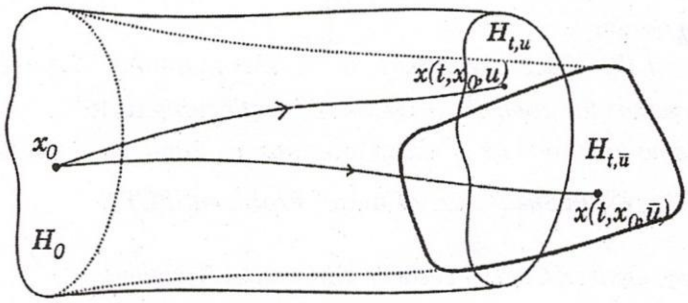

Fig. 1. The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP).

If $\mu(H_{t,u})$ is a measure of the set $H_{t,u}$ then $\mu(H_{t,u})$ is called a cross-area of sheaf trajectory at (t,u), in particular it is a square of set $H_{t,u}$.That is $\mu(H_{t,u}) = \int_{H_{t,u}} dx_t$ and

$\mu(H_{t,\overline{u}}) = \int_{H_{t,\overline{u}}} d\overline{x}_t$ is a square of $H_{t,\overline{u}}$.

Assumption 1. Suppose that the vector function f (t,x(t),u(t)) satisfies

i) $$\left\| \frac{\partial f}{\partial x} \Delta x(t) + \frac{\partial f}{\partial u} \Delta u(t) \right\| \le M(\|\Delta x(t)\| + \|\Delta u(t)\|) \quad (3)$$

ii) $$\sum_{k=2}^{+\infty} \frac{1}{k!} \left\| d^k f \right\| \le m \quad (4)$$

iii) $$\left| sp \frac{\partial f(t, x(t, x_0, u(t)), u(t))}{\partial x} \right| = L(\|u(t)\|) \quad (5)$$

for all $x(t) \in Q \subset R^n$, $u(t), \overline{u}(t) \in U \subset E^p$, $t \in I$, where M, m, L are real positive constants and spA is trace of matrix A.

Lemma 1. *For the fuzzy controls* u(t) *and* $\overline{u}(t) \in U \subset E^p$, *the norm of* $\Delta u = \overline{u}(t) - u(t)$ *is estimated as follows:*

a) $$\| \Delta u \|_C \le \sqrt{p} \quad (6)$$

b) $\|\Delta u\|_L = \int_0^T \|\Delta u(t)\| dt \le T\sqrt{p}$, (7)

Proof of Lemma 1: Let $u(t), \bar{u}(t) \in U \subset E^p$ are fuzzy controls. In [5], we defined a fuzzy function $u: I \to U \subset E^p = E \times E \times ... \times E$, that means $u(t) = (u_1(t), u_2(t), ..., u_p(t))$. Because every $u_k(t)$ satisfies $|u_k(t)| \le 1$ (k=1,2,..p) then a norm of

a) $\|\Delta u\|_C = \max\{\|\bar{u}(t) - u(t)\| : t \in I\}$

$$\le \max\left\{\sqrt{\sum_{i=1}^{p} |\bar{u}_i(t) - u_i(t)|^2} \ : t \in I\right\} \le \sqrt{p}$$

where $u(t), \bar{u}(t) \in U \subset E^p$

b) $\|\Delta u\|_L = \int_0^T \|\Delta u(t)\| dt \le \sqrt{p} \int_0^T dt \le T\sqrt{p}$ (■)

Theorem 1. *Suppose that* $u(t), \bar{u}(t) \in U \subset E^p$ *are fuzzy controls. If the function* $f(t, x(t), u(t))$ *satisfies* (3) *and* (4) *then the norm of* $\Delta x = x(t, x_0, \bar{u}(t)) - x(t, x_0, u(t))$ *is estimated as follows:*

a) $\|\Delta x\|_C \le (Tm + M\sqrt{p}) \exp(MT)$ (8)

b) $\|\Delta x\|_L \le T^2 (m + M\sqrt{p}) \exp(MT)$ (9)

Proof of Theorem 1: Let $u(t), \bar{u}(t) \in U \subset E^p$ are fuzzy controls with $\Delta u = \bar{u}(t) - u(t)$ satisfies (6) or (7) .

a) The solutions of (1) are equivalent the following integrals:

$$x(t) = x_0 + \int_0^t f(s, x(s), u(s)) ds \quad \text{and} \quad \bar{x}(t) = x_0 + \int_0^t f(s, \bar{x}(s), \bar{u}(s)) ds.$$

Estimating $\|\Delta x(t)\|$ as follows $\|\Delta x(t)\| \le \int_0^t \|f(s, \bar{x}(s), \bar{u}(s) - f(s, x(s), u(s))\| ds$

$$\le \int_0^t \left\| \frac{\partial f}{\partial x}(s, x(s), \bar{u}(s)) dx + \frac{\partial f}{\partial u}(s, x(s), u(s)) du + \sum_{k=2} d^k f(s, x(s), u(s)) \right\| ds$$

$$\le M \int_0^t \|dx + du + ..\| ds \le M \int_0^t \|\Delta x(s)\| ds + M \int_0^t \|\Delta u(s)\| ds + mT$$

$$\le M \int_0^t \|\Delta x(s)\| ds + MT\sqrt{p} + mT$$

By Gronwall-Bellmann's Lemma, it implies that

$$\|\Delta x\|_C = \max_{t \in [0,T]} \|\Delta x(t)\| \le T(m + M\sqrt{p}) \exp(MT)$$

b) $$\|\Delta x(t)\| \le M \int_0^t \|\Delta x(s)\| ds + M \int_0^t \|\Delta u(s)\| ds + mT$$

$$\|\Delta x(t)\| \le M \int_0^t \|\Delta x(s)\| ds + MT\sqrt{p} + mT$$

$$\le T(m + M\sqrt{p}) \exp(MT)$$

For $\quad \|\Delta x\|_L = \int_0^T \|\Delta x(t)\| dt \leq T^2 (M\sqrt{p} + m)\exp(MT)$ we have (9) (■)

Theorem 2. *Suppose that* $u(t), \overline{u}(t) \in U \subset E^p$ *are fuzzy controls, if the function* $f(t, x(t), u(t))$ *satisfies* (3) *and* (4) *then the norm of*

$$\Delta f = f(t, x(t, x_0, \overline{u}(t)), \overline{u}(t)) - f(t, x(t, x_0, u(t)), u(t))$$

is *estimated as follows:*

a) $\|\Delta f\|_C \leq MT[(M\sqrt{p} + m)\exp(MT) + \sqrt{p}] + m$ (10)

b) $\|\Delta f\|_L \leq T\left\{M\left[T(m + M\sqrt{p})\exp(MT) + \sqrt{p}\right] + m\right\}$ (11)

Proof of Theorem 2:

a) For $\leq \max\left\{\left\| df + \frac{1}{2!}d^2 f + \frac{1}{3!}d^3 f + \ldots \right\| : t \in I\right\}$

$$\leq \max\left\{\|df\| + \sum_{k=2}^{+\infty}\frac{1}{k!}\|d^k f\| : t \in I\right\}$$

$$\leq \max\left\{\left\|\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}du\right\| + \sum_{k=2}^{+\infty}\frac{1}{k!}\|d^k f\| : t \in I\right\}$$

$$\leq M(\|\Delta x\|_C + \|\Delta u\|_C) + m$$

$$\leq M[T(M\sqrt{p} + m)\exp(MT) + T\sqrt{p}] + m$$

$$\leq MT[(M\sqrt{p} + m)\exp(MT) + \sqrt{p}] + m$$

b) For $\quad \|\Delta f\|_L = \int_0^T \|f(s, \overline{x}(s, x_0, \overline{u}(s)), \overline{u}(s)) - f(s, x(s, x_0, u(s)), u(s))\| ds$

$$\leq M\left(\int_0^T \|\Delta x(t)\| dt + \int_0^T \|\Delta u(t)\| dt\right) + m\int_0^T dt$$

$$\leq M(\|\Delta x\|_L + \|\Delta u\|_L) + mT$$

$$\leq M\left[T^2(m + M\sqrt{p})\exp(MT) + T\sqrt{p}\right] + mT$$

$$\leq T\left\{M\left[T(m + M\sqrt{p})\exp(MT) + \sqrt{p}\right] + m\right\}$$ (■)

3. THE COMPARISON OF SHEAF SOLUTIONS IN THE SFCP

Lemma 2. *For* $A, B \geq 0$ *there exists a real number K such that* $e^A - e^B \leq K e^{A-B}$.

Proof of Lemma 2: We have $e^A - e^B = e^B(e^{A-B} - 1) \leq K e^{A-B}$, $K > e^B$ (■)

Now, suppose that $\mu(H_0)$ is given. There are many following results of comparison of sheaf- solutions :

Theorem 3. *Suppose that* $u(t), \overline{u}(t) \in U \subset E^p$ *are fuzzy controls. If the function* $f(t, x(t), u(t))$ *satisfies* (3) ,(4) *and* (5) *then we have the following estimation:*

$$|\mu(H_{T,\overline{u}}) - \mu(H_{T,u})| \leq \mu(H_0)\exp(LT\sqrt{p}) \quad (12)$$

Proof of Theorem 3: We have $\mu(H_{t,u}) = \int_{H_{t,u}} dx_t = \int_{H_0} \left| \det \frac{\partial x(t,x_0,u)}{\partial x_0} \right| dx_0$,

where

$$\left| \det \frac{\partial x(t,x_0,u)}{\partial x_0} \right| = \exp\left(\int_0^T sp \frac{\partial f(\gamma, x(\gamma,x_0,u), u(\gamma)}{\partial x} d\gamma \right) ,$$

that means

$$\|\Delta f\|_C = \max\left\{ \| f(t, x(t,x_0,\bar{u}(t)), \bar{u}(t)) - f(t, x(t,x_0,u(t)), u(t)) \| : t \in I \right\}$$

$$\mu(H_{t,u}) = \int_{H_0} \left| \det \frac{\partial x(t,x_0,u)}{\partial x_0} \right| dx_0 = \int_{H_0} \left| \exp\left(\int_0^T sp \frac{\partial f(\gamma, x(\gamma,x_0,u), u(\gamma))}{\partial x} d\gamma \right) \right| dx_0$$

$$\mu(H_{T,u}) = \mu(H_0)\exp(L\int_0^T \|u(t)\| dt) .$$

It is analogous of proof a) above, we have $\mu(H_{T,u}) = \mu(H_0)\exp(L\int_0^T \|\bar{u}(t)\| dt)$.

Estimating $| \mu(H_{T,\bar{u}}) - \mu(H_{T,u}) |$ we have

$$| \mu(H_{T,\bar{u}}) - \mu(H_{T,u}) | \le \mu(H_0) \left[\exp(L\int_0^T \|\bar{u}(t)\| dt) - \exp(L\int_0^T \|u(t)\| dt) \right]$$

$$\le \mu(H_0) K \exp[L\int_0^T (\|\bar{u}(t)\| - \|u(t)\|) dt]$$

$$\le \mu(H_0) K \exp[L\int_0^T \|\Delta u(t)\| dt] \le \mu(H_0) K \exp[LT\sqrt{p}]$$

where $K \ge \exp(LT\sqrt{p})$. (■)

Corollary 1 *Suppose that* $u(t), \bar{u}(t) \in U \subset E^p$ *are fuzzy controls. If the function* $f(t, x(t), u(t))$ *satisfies* (3) *and* (4), *then for* (1) *when n=1 we have the following estimation:*

$$| \mu(H_{T,\bar{u}}) - \mu(H_{T,u}) | \le (b_0 - a_0)\exp(2LT\sqrt{p}), \tag{13}$$

where $K = \exp(LT\sqrt{p})$.

Proof of Corollary: When $n = 1$ we have $\mu(H_0) = b_0 - a_0$, finally we get (13) (see fig.2) .

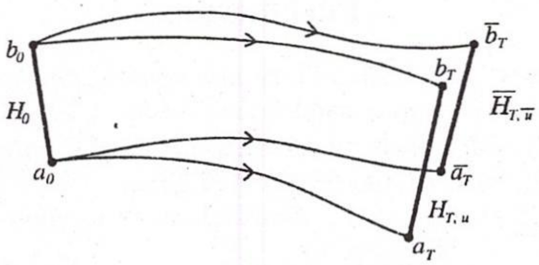

Fig. 2. The sheaf-solutions of Sheaf Fuzzy Control Problem (SFCP), when n = 1. (■)

4. CONCLUSION

In the Sheaf Fuzzy Control Problem (SFCP) for many different fuzzy controls $u(t), \bar{u}(t) \in U \subset E^p$ we have the comparison (7)-(13).There are differences between the Sheaf

Fuzzy Control Problem (SFCP) and the Sheaf Optimal Control Problem in Fuzzy Type (SOFCP) what was offered in [5].

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SO SÁNH BÓ NGHIỆM TRONG BÀI TOÁN ĐIỀU KHIỂN MỜ

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TÓM TẮT: *Trong* [2] *tác giả đã xét bài toán điều khiển tối ưu bó (SOCP) cho bởi hệ phương trình vi phân:*

$$\frac{dx(t)}{dt} = f(t, x(t), u(t))$$

ở đây $x_0 \in Q \subset R^n, u \in U \subset R^p, t \in [0,T] \subset R^+$, *và bó nghiệm:*

$$H_{t,u} = \{x(t) = x(t, x_0, u(t)) \mid x_0 \in H_0 \subseteq Q, t \in I = [0,T], u(t) \in U\}$$

với hàm mục tiêu $I(u) \to \min$.

Trong [5] *lại trình bày các điều kiện cần của* ***bài toán điều khiển tối ưu bó dạng mờ*** *(SOFCP), với các điều khiển mờ* $u(t) \in U \subset E^p$ *thay vì thuộc* R^p.

Bài báo này đưa ra các so sánh các bó nghiệm $H_{t,u}$ *và* $H_{t,\bar{u}}$ *ứng với các điều khiển mờ khác nhau* $u(t), \bar{u}(t) \in U \subset E^p$ *của* ***bài toán điều khiển bó dạng mờ*** *(SFCP).*

Từ khóa: *Lý thuyết mờ, Lý thuyết điều khiển tối ưu, Phương trình Vi phân*

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