FINITE ELEMENT ANALYSIS OF ELASTO-PLASTIC BOUNDARY FOR SOME STRUCTURE PROBLEMS

Trương Tích Thiện (1), **Cao Bá Hoàng** (2)
(1) Trường Đại học Bách Khoa, ĐHQG-HCM
(2) Bộ Xây Dựng

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ABSTRACT: *The finite element method (FEM) is used widely in analysis of elasto-plastic behaviours for structures. The analysis often involves a two-stage process: first, the internal force field acting on the structural material must be defined, and second, the response of the material to that force field must be determined. In other words, the analysis of behaviours of structural material is establishment relationships between stresses and strains in the structure in the plastic as well as elastic ranges. It furnishes more realistic estimates of load-carrying capacities of structures and provides a better understanding of the reaction of the structural elements to the forces induced in the material.*

Key words: Elasto-plastic, plasticity, Timoshenko, analysis

1. INTRODUCTION

It is generally regarded that the origin of plasticity, as a branch of mechanics of continua, dated back to a series of papers from 1864 to 1872 by Tresca on the extrusion of metals, in which he proposed the first yield condition. The actual formulation of the theory was done in 1870 by St. Venant, who introduced the basic constitutive relations for what today we would call rigid, perfectly plastic materials in plane stress. A generalization similar to the results of Levy was arrived independently by von Mises in a landmark paper in 1913, accompanied by his well-known, pressure-insensitive yield criterion (J2-theory, or octahedral shear stress yield condition).

In 1924, Prandtl extended the St. Venant-Levy-von Mises equations for the plane continuum problem to include the elastic component of strain, and Reuss in 1930 carried out their extension to three dimensions. The appropriate flow rule associated with the Tresca yield condition, which contains singular regimes (i.e., corners or discontinuities in derivatives with respect to stress), was discussed by Reuss in 1932 and 1933 [1].

In 1958, Prager further extended this general framework to include thermal effects (non-isothermal plastic deformation), by allowing the yield surface to change its shape with temperature.A very significant concept of work hardening, termed the material stability postulate, was proposed by Drucker in 1951 and amplified in his further papers. With this concept, the plastic stress-strain relations together with many related fundamental aspects of the subject may be treated in a unified manner [1]

2. FINITE ELEMENT ANALYSIS OF ELASTO-PLASTIC BOUNDARY

2.1.Formulation of the elasto-plastic matrix: *3-D elasto-plastic stiffness matrix*

The equation of the incremental stress-strain relation as follows [1]:

$$d\sigma ij = C^{ep}_{ijkl}\, d\varepsilon_{kl} = (C_{ijkl} + C^{p}_{ijkl})d\varepsilon_{kl} \tag{1}$$

in which the incremental stress and strain tensors $d\sigma_{ij}$, $d\varepsilon_{ij}$ are generally expressed in vector forms:

$$\{d\sigma\}^T = \{d\sigma_x\ d\sigma_y\ d\sigma_z\ d\tau_{yz}\ d\tau_{zx}\ d\tau_{xy}\ \} \tag{2}$$

$$\{d\varepsilon\}^T = \{d\varepsilon_x\ d\varepsilon_y\ d\varepsilon_z\ d\gamma_{yz}\ d\gamma_{zx}\ d\gamma_{xy}\ \} \tag{3}$$

and C_{ijkl} is the tensor of elastic modulus expressed in matrix form:

$$[C]=\begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K+\frac{3}{4}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{3}{4}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{4}$$

where G and K are the shear and bulk moduli, respectively.

$$G=\frac{E}{2(1+\nu)} \text{ and } K=\frac{E}{3(1-2\nu)} \tag{5}$$

(E is Young's modulus and ν Poisson's ratio)

C^{p}_{ijkl} is the plastic stiffness tensor.

$$[Cp]=-\frac{1}{H}\begin{bmatrix} \bar{S}_x^2 & & & & & \text{symmetric} \\ \bar{S}_y\bar{S}_x & \bar{S}_y^2 & & & & \\ \bar{S}_z\bar{S}_x & \bar{S}_z\bar{S}_y & \bar{S}_z^2 & & & \\ \bar{S}_{yz}\bar{S}_x & \bar{S}_{yz}\bar{S}_y & \bar{S}_{yz}\bar{S}_z & \bar{S}_{yz}^2 & & \\ \bar{S}_{zx}\bar{S}_x & \bar{S}_{zx}\bar{S}_y & \bar{S}_{zx}\bar{S}_z & \bar{S}_{zx}\bar{S}_{yx} & \bar{S}_{zx}^2 & \\ \bar{S}_{xy}\bar{S}_x & \bar{S}_{xy}\bar{S}_y & \bar{S}_{xy}\bar{S}_z & \bar{S}_{xy}\bar{S}_{yz} & \bar{S}_{xy}\bar{S}_{zx} & \bar{S}_{xy}^2 \end{bmatrix} \tag{6}$$

in which $$\frac{1}{H}=\frac{36G^2}{h} \tag{7}$$

2.2. Elasto-plastic Timoshenko beam analysis

2.2.1. Timoshenko beam theory

This theory allows for transverse shear deformation effects while Euler-Bernoulli beam theory takes no account of transverse shear deformation.

The governing equation: $[Kf+Ks]\varphi-f=0$ (8)

where, the submatrices of Kf and Ks and subvectors of f for element e.

Element stiffness matrix by using a 1-point Gauss-Legendre rule:

$$K_f^{(e)}=\left(\frac{EI}{l}\right)^{(e)}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{9}$$

$K_s^{(e)}$ is evaluated exactly using a 2-point Gauss-Legendre rule:

$$K_s^{(e)}=\left(\frac{GA}{l}\right)^{(e)}\begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1^2}{3} & -\frac{1}{2} & \frac{1^2}{6} \\ -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{l}{2} & \frac{1^2}{2} & -\frac{1}{2} & \frac{1^2}{3} \end{bmatrix}^{(e)} \tag{10}$$

2.2.2.Elasto-plastic layered Timoshenko beams

Formulations in the layer approach

Bending moment M and shear force Q by using the mid-ordinate rule:

$$M = EI\left(-\frac{d\theta}{dx}\right)\ ; \qquad Q = G\,A\varepsilon_s \qquad (12)\text{-}(13)$$

where

$$EI = \sum_l E_l(b_l z_l^2 t_l)\ ;\ \ GA = \sum_l G_l b_l t_l \qquad (14)\text{-}(15)$$

in which bl is the layer breadth, tl is the layer thickness, zl is the z-coordinate at the middle of the layer, El is Young's modulus of the layer material, Gl is the shear modulus of the layer material.

If the stress at the middle surface of a layer reaches the uniaxial yield stress of the layer material, the whole layer is considered to be plastic and El is replaced by $E_l\left(1-\frac{E_l}{E_l+H'}\right)$; where H' is the uniaxial strain hardening parameter.

3.BEAM PROBLEM

Finite element idealisation:

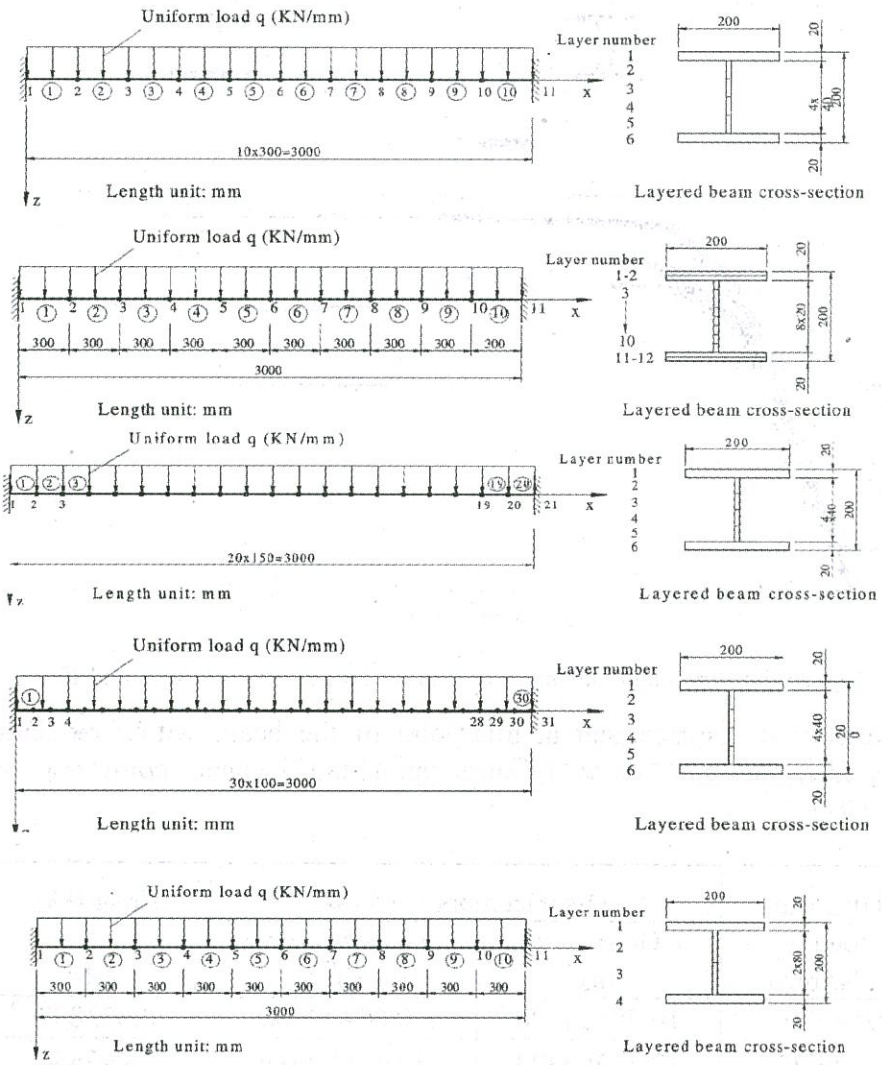


Fig. 1. Finite element idealisation of meshes M1, M2, M3, M4, M5

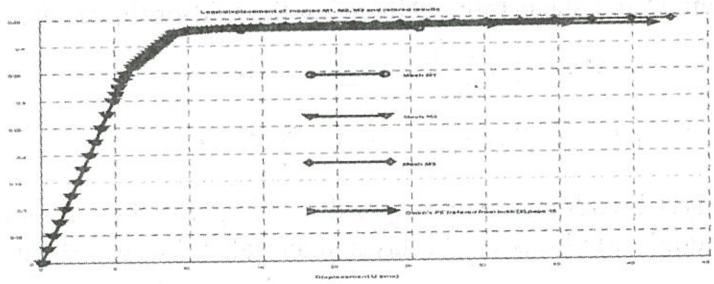
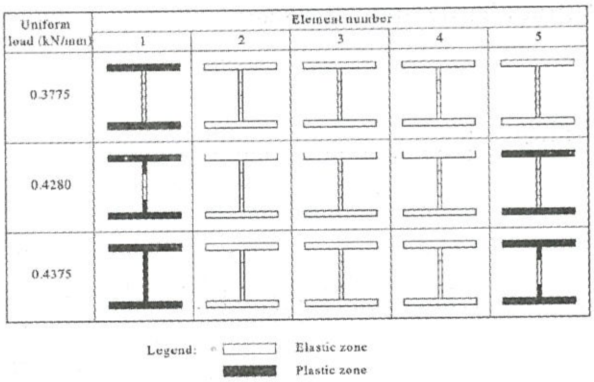


Fig. 2. Uniform load – displacement curves for meshes M_1, M_2, M_3 and Owen's FE

Table 1. Distribution of plastic layers of some sections at elements with various uniform load of mesh M2



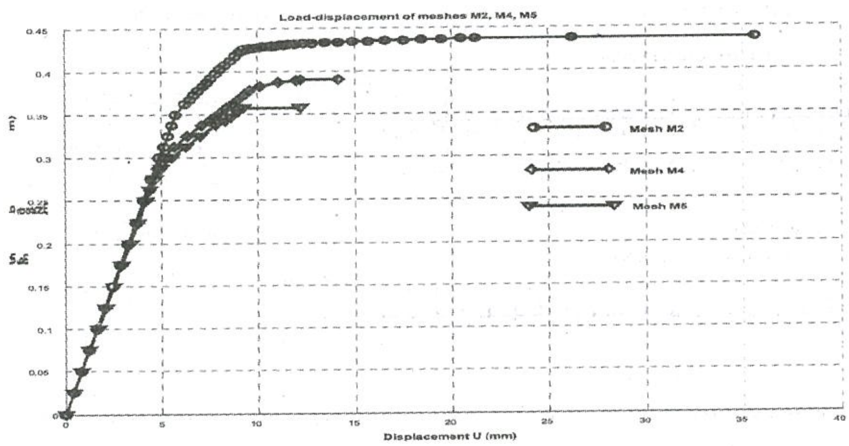


Fig. 3. Uniform load – displacement curves for meshes M2, M4 and M5

Table2. Comparison of displacement at mid-point of the beam with formulation of shear stiffness matrix *[Ks]* computed with 1-Gauss point and 2-Gauss point rule of mesh M2 (tolerance $\varepsilon_D = 10^{-3}$)

Uniform load q (kN/mm)	Displacement U (mm)		Error (%)
	1-Gauss point (a)	2-Gauss point (b)	a-b
0.4290	10.70154906	10.63478666	0.623857
0.4320	12.31206323	12.13690619	1.422646

0.4325	12.83619912	12.57407891	2.042039
0.4365	19.45118787	18.37697980	5.522583
0.4370	20.46772651	19.30164554	5.697169

The Timoshenko beam theory has got a difficulty by using the shear stiffness matrix *[Ks]* because it may lead to "locking" phenomenon with 2-point Gauss-Legendre rule formulation.

4. PLANE STRAIN AND AXISYMMETRIC PROBLEMS IN SOLID MECHANICS APPLICATIONS

4.1. Problem description: Thick-walled cylinder under internal pressure problem

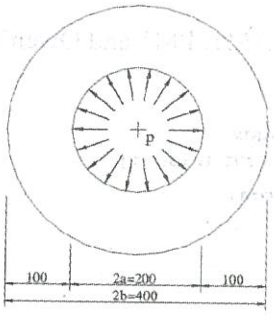


Fig. 4. A thick-walled cylinder under

Material properties:

Elastic modulus: $E = 2.1e4\ dN/mm^2$

Poissons ratio: $\nu = 0.3$

Uniaxial yield stress: $\sigma_y = 24.0\ dN/mm^2$

Strain hardening parameter: $H' = 0.0$

Geometry proportions:

Internal radius: $a = 100\ mm$

External radius: $b = 200\ mm$

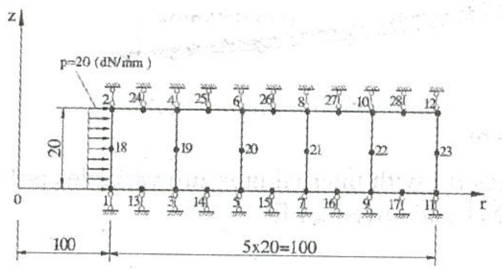


Fig. 5. Finite element idealisation of axisymmetric problem, mesh AM1

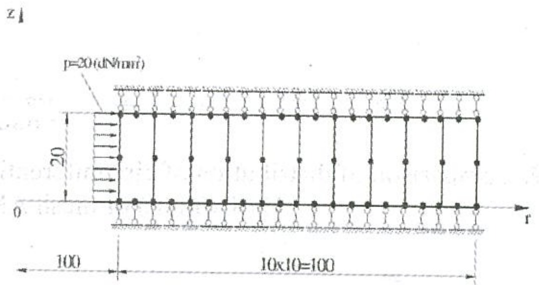


Fig. 6. Finite element idealisation of axisymmetric problem, mesh AM2

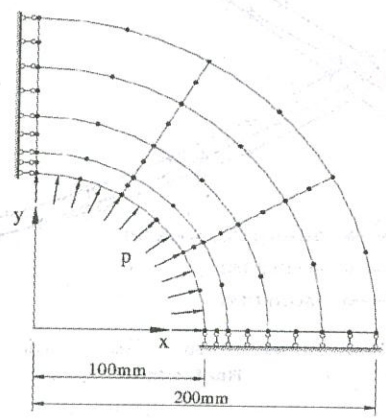


Fig. 7. Finite element idealisation of plane strain problem, mesh PM1

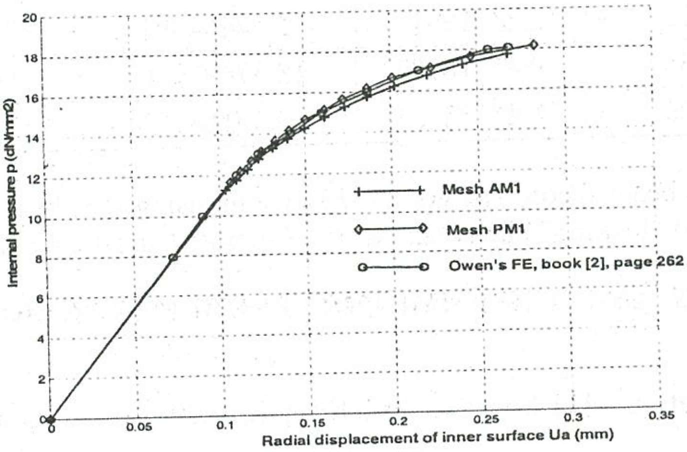


Fig. 8. Radial displacement Ua(mm) of inner face of Mesh AM1, PM1 and Owen's FE

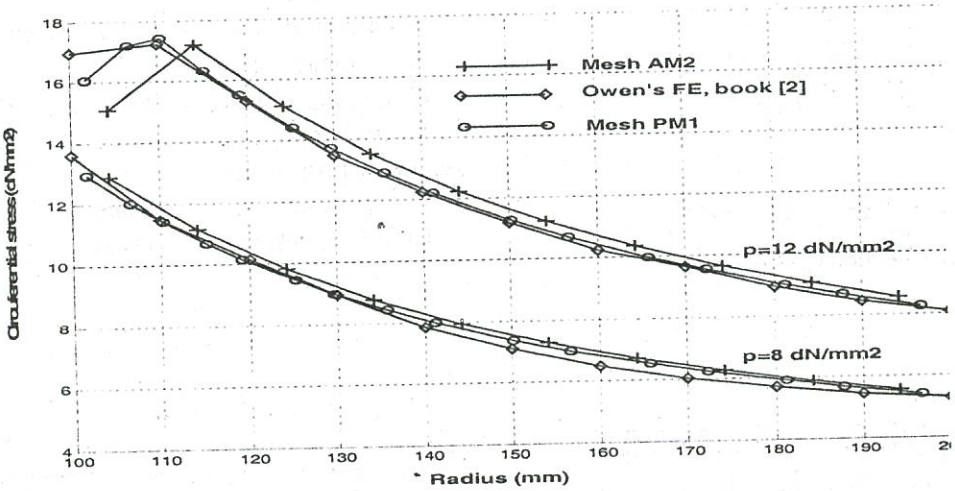


Fig. 9. Comparison of distribution of circumferential stress σ_θ with internal pressure variables p=8 and 12(dN/mm^2) of mesh AM2, PM1 and Owen's FE.

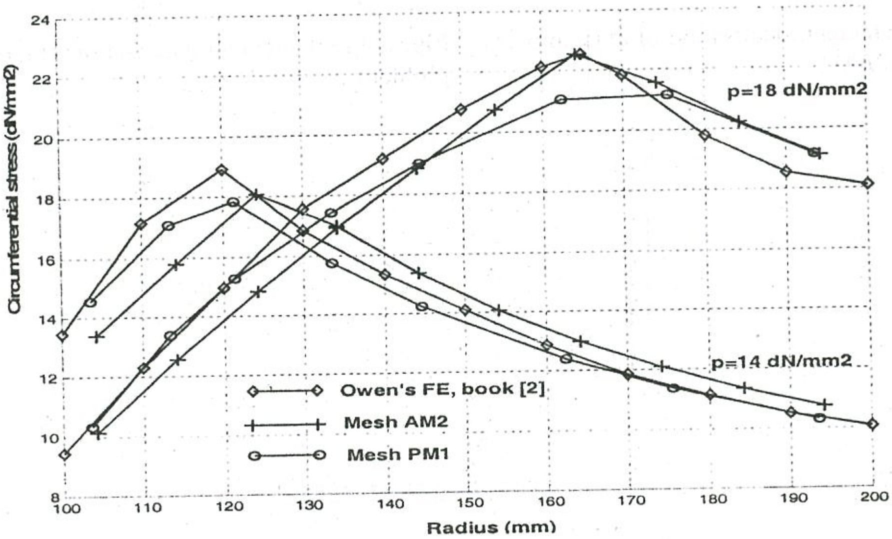


Fig. 10. Comparison of distribution of circumferential stress σ_θ with internal pressure variables p=14 and 18(dN/mm^2) of mesh AM2, PM1 and Owen's FE

5. CONCLUSION

For the Timoshenko beam problem, the analysis of elasto-plastic behaviour of the beam considered development of plastic zone in beam sections through determining plastic layers. However, the Timoshenko beam theory has met a difficulty by using the shear stiffness matrix *[Ks]* because it may lead to "locking" phenomenon with 2-point Gauss-Legendre rule formulation. This phenomenon can be cured by using 1-point Gauss-Legendre rule formulation for the shear stiffness matrix. The obtained solutions are sensitive with meshes. The more number of layers is the more stiffness of the beam. Unfortunately, the experimental results are not available to compare with the obtained solutions by this approach.

For the considered 2-D problem, the results obtained from the present FE of several meshes, even for coarse mesh, is close. However, the obtained results of meshes of the axisymmetric problem model are different with the results obtained by the plane strain problem model. The variation stress was rather smooth without concentration of stress.

The modelisation of axisymmetric problem with each element having differential stiffness matrix is especially adaptive for analyzing some thick-walled pipes structures made by composite material! Elements containing differential material properties have differential stiffness or they have differential stiffness matrix.

Application of the models can be used to analyse elasto-plastic behaviour for some thick-walled pipes made by composite materials (especially reinforced concrete pipes) and "sandwich" materials.

PHƯƠNG PHÁP PHẦN TỬ HỮU HẠN TRONG PHÂN TÍCH GIỚI HẠN ĐÀN HỒI - DẺO CỦA MỘT SỐ BÀI TOÁN CẤU TRÚC

Truong Tich Thien[(1)], Cao Ba Hoang[(2)]
(1) University of Technology, VNU-HCM
(2) Ministry of Construction

TÓM TẮT: *Phương pháp phần tử hữu hạn được sử dụng rộng rãi trong việc phân tích ứng xử đàn–dẻocủa các cấu trúc. Việc phân tích thường bao gồm quá trình hai giai đoạn: xác định trường nội lực tác động lên vật liệu cấu trúc và đáp ứng của vật liệu ứng với trường nội lực đó. Nói cách khác, việc phân tích các ứng xử của cấu trúc là sự thiết lập những mối quan hệ giữa ứng suất và biến dạng trong cấu trúc biến dạng dẻo cũng như đàn hồi. Nó đưa đến những đánh giá thực hơn các khả năng chịu tải của các cấu trúc và cung cấp sự hiểu biết tốt hơn về phản ứng của các phần tử kết cấu đối với những nội lực bên trong vật liệu.*

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