SHEAF- OPTIMAL CONTROL PROBLEM IN FUZZY TYPE

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ABSTRACT: *One of the main advantages of fuzzy theory is over more conventional in solving complex nonlinear control problems. Besides, fuzzy schemes can be used as self-reliant control solutions. In this paper, we woud like to show the Sheaf Optimal Control Problem (SOCP) that means optimal control of many processes(on sheaf) and Sheaf Optimal Control in Fuzzy type (SOCPF).*

Keywords: *Fuzzy Theory; Optimal Control Theory; Differential Equations*

1. INTRODUCTION

The optimal control problem of differential equations has been studying in both of theory and application (see [1, 3, 4, 6, 10, 11, 14, 15]). In this paper, we study the so called sheaf optimal control and apply fuzzy controls to SOCP (which is called sheaf optimal control in fuzzy type (SOCPF)).

The paper is organized as follows: In the second section, some basic concepts and notations which are useful in the next sections are recalled. In section 3, the Sheaf-Optimal Control Problem (SOCP) is formulated, the dependence of its sheaf solutions on given controls is studies and many auxiliary lemmas for the following main directions of investigation are given. Some necessary conditions for SOCP are studied in section 4.

2. PRELIMINARIES

In [13], we presented how to fuzzy a crisp set A to be a fuzzy control set U. Some concepts in [7, 9] in the appropriate form to be used later are recalled.

Definition 1

Let us denote by $E^1 = \{u : R \supset I \to [0,1] \text{ such that } u \text{ satisfies (i) to (iv) mentioned below}\}$

(i) u is normal, that is, there exists a $t_0 \in I$ such that $u(t_0) = 1$;

(ii) u is fuzzy convex, that is , for $t_1, t_2 \in I$ and $0 \le \lambda \le 1$,

$$u(\lambda t_1 + (1-\lambda)t_2) \ge \min\{u(t_1), u(t_2)\};$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = cl\{t \in I : u(t) > 0\}$ is compact.

Next, we use some norms as follows: with $u : I \to U \subset R^p$, $\|u\|_L = \int_0^T \|u(t)\| dt$ and with the mapping $x : I \to R^n$, $\|x\|_C = \max\{\|x(t)\|, t \in I\}$, where $\|u(t)\|, \|x(t)\|$ are Euclidean norm in R^p, R^n, respectively.

3. THE SHEAF OPTIMAL CONTROL PROBLEM

A system of differential equations is considered

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \tag{1}$$

where $x(0) = x_0 \in H_0 \subset R^n$, $u(t) \in U \subset R^p$, $t \in I = [0,T] \subset R^+$ and $f: I \times R^n \times R^p \to R^n$.

Definition 2

By the sheaf solution of (1) we mean the set of its solutions which give at the time t a set $H_{t,u} = \{x(t) = x(t, x_0, u) - \text{solution of(1)} \mid x_0 \in H_0\}$, where $u(t) \in U$, $t \in I$. (2)

The set $H_{t,u}$ is called the *cross-area* at (t,u) of the above sheaf solution.

Definition 3

Problem of solving (1) and on the set (2) satisfying

$$J(u) \to \min, \tag{3}$$

where J(u) - a goal (cost) funcion defined in section 4, is called the Sheaf Optimal Control Problem (SOCP).

In the case, when a control u(t) is fuzzy, Sheaf Optimal Control Problem in Fuzzy Type (SOCPF) is given.

In the next section, some types of goal function J(u) could be studied.

Definition 4

A function $u(t) \in U$ is called an admissible control if it is bounded and (Lebesgue) measurable.

Suppose at time $t = 0, u(0) = 0$ and $x(0) = x_0 \in H_0$. For two admissible controls $u(t)$ and $\bar{u}(t) = u(t) + \Delta u(t) \in U$, where $\Delta u(t) = \bar{u}(t) - u(t)$, two sets of sheaf-solutions are given:

$$H_{t,u} = \{x(t) = x(t, x_0, u) - \text{solution of(1)} \mid x_0 \in H_0\}$$
$$H_{t,\bar{u}} = \{\bar{x}(t) = x(t, x_0, \bar{u}) - \text{solution of(1)} \mid x_0 \in H_0\}$$

where $t \in I, u(t), \bar{u}(t) \in U$. (See fig.1)

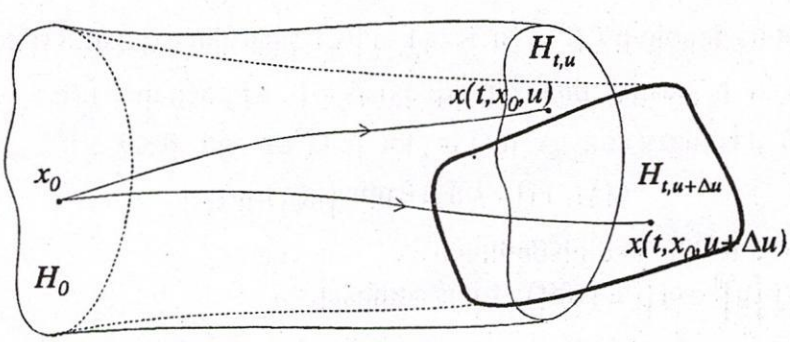

Fig.1

As we know, the solutions of differential equations depend locally on initial conditions, right hand side and parameters. From now on, we suppose that system (1) is controlable, has solutions and H_0, U are bounded. In what below, the solution x(t) and right hand side function f depend on variation of control u(t) for all $x_0 \in H_0$ wiil be proved

We often use the assumption as follows.

Assumption 1. Vector function f satisfies

$$\left\|f(t,x(t),u(t))-f(t,\bar{x}(t),\bar{u}(t))\right\| \le c(t)\left[\left\|\bar{x}(t)-x(t)\right\|+\left\|\bar{u}(t)-u(t)\right\|\right],$$

for all $t \in I$; $x(t),\bar{x}(t) \in R^n$; $u(t),\bar{u}(t) \in U$, where c(t) is positive and integrable on I.

Let $C = \int_0^T c(t)\,dt$. Because c(t) is integrable on I, it is bounded almost everywhere by a positive constant M.

Lemma 1: *Suppose that the vector function f(t,x(t),u(t)) is continuous in t and satisfies Assumption 1. For every* $\varepsilon > 0$ *and any choices of* $x_0 \in H_0, t \in I$, *there exists a* $\delta(\varepsilon) > 0$ *such that* $\|\Delta x\|_C \le \varepsilon$ *if* $\|\Delta u\|_L < \delta(\varepsilon)$.

Proof. The solutions of (1) for controls u(t) and $\bar{u}(t)=u(t)+\Delta u(t)$ are equivalent to the following integral forms:

$$x(t)=x_0+\int_0^t f(s,x(s),u(s))ds \quad \text{and}$$

$$\bar{x}(t)=x_0+\int_0^t f(s,\bar{x}(s),\bar{u}(s))ds.$$

Estimating $\Delta x(t)=\bar{x}(t)-x(t)$ by norm

$$\|\Delta x(t)\| \le \int_0^t \left\|f(s,\bar{x}(s),\bar{u}(s))-f(s,x(s),u(s))\right\|ds.$$

By the Assumption 1, one has:

$$\|\Delta x(t)\| \le \int_0^t c(s)\left[\|\Delta x(s)\|+\|\Delta u(s)\|\right]ds$$

$$\le \int_0^t c(s)\|\Delta x(s)\|ds + M\int_0^t \|\Delta u(s)\|ds. \tag{4}$$

Then using $\|\Delta u\|_L < \delta(\varepsilon)$, the follow is obtained:

$$\|\Delta x(t)\| \le \int_0^t c(s)\|\Delta x(s)\|ds + M\delta(\varepsilon).$$

Using the Gronwall-Bellman's Lemma (sec [10, 12]), we have the following

$$\|\Delta x(t)\| \le M\delta(\varepsilon).\exp(MT).$$

If we choose $\quad 0<\delta(\varepsilon)\le \varepsilon / M\exp(MT)$, (5)

then, it follows $\|\Delta x\|_C = \max_{t\in[0,T]} \|\Delta x(t)\| \le \varepsilon$.

Lemma 2: *Suppose that the vector function f(t,x(t),u(t)) is continuous in t and satisfies Assumption 1. For every* $\varepsilon > 0$ *and any choices of* $x_0 \in H_0, t \in I$, *there exists a* $\delta(\varepsilon) > 0$ *such that* $\int_0^T \left\|f(s,\bar{x}(s),\bar{u}(s))-f(s,x(s),u(s))\right\|ds \le \varepsilon$ *if* $\|\Delta u\|_L < \delta(\varepsilon)$.

The proof of Lemma 2 is deduced from that of Lemma 1.

Definition 5

By a fuzzy control *u(t)* of (1) we mean a function

$$u: I \to U \subset E^p = E^1 \times E^1 \times \ldots \times E^1 \ (p-\text{times})$$

where $u(t)=(u_1(t),\ldots,u_p(t)) \in E^p$.

Given two fuzzy controls u and $\bar{u}$, Δu is not a fuzzy control, in general. From now on, it is assumed that if u and $\bar{u}$ are fuzzy then Δu is fuzzy.

When u is a fuzzy control, the norm of Δu is estimated as follows.

Lemma 3. *If u(t) and* $\bar{u}(t)$ *are fuzzy controls, then* $\|\Delta u\|_L \le T\sqrt{p}$.

Proof. The norm of Δu is computed:

$$\|\Delta u\|_L = \int_0^T \|\Delta u(t)\| dt = \int_0^T \left(\sum_{i=1}^{p} (\Delta u_i(t))^2 \right)^{\frac{1}{2}} dt \le T\sqrt{p} .$$

Here, we have employed the property of the fuzzy control $u(t) = (u_1(t), ..., u_p(t))$, where $u_i(t) \in E^1, i = 1, 2, ..., p$.

Theorem1. *Suppose that the vector function f(t,x(t),u(t)) is continuous in t and satisfiesAssumption 1. For every controls* u(t), $\bar{u}$ (t) *and any choices of* $x_0 \in H_0, t \in I$, *one has* $\|\Delta x\|_C \le MT\sqrt{p} \exp(C)$. (6)

Proof. The proof is analogue of that of Lemma 1. From (4) and the norm of a fuzzy control in Lemma 3, we estimate

$$\|\Delta x(t)\| \le \int_0^t c(s) \|\Delta x(s)\| ds + MT\sqrt{p}$$

Using the Gronwall-Bellman's Lemma (see [10, 12]), we get

$\|\Delta x(t)\| \le MT\sqrt{p} \exp(C)$. It implies (6). (7)

It follows from Lemma 1 and 3 the below theorem.

Theorem2. *Suppose that the vector function f(t,x(t),u(t)) is continuous in t and satisfies Assumption 1. For every controls* u(t), $\bar{u}$ (t) *and any choices of* $x_0 \in H_0, t \in I$, *one has* $\int_0^T \|f(s, \bar{x}(s), \bar{u}(s)) - f(s, x(s), u(s))\| ds \le MT\sqrt{p} \exp(C)$.

4. NECESSARY CONDITIONS FOR THE SOCP

In this section, some types of the goal (cost) function J(u) are considered. We will denote by A^T the transpose of the matrix A. Notation $0 \in R^p$ means that the p-dimensional zero vector in space R^p. U is open set in R^p.

Given two functions $G \in C^1\left[R^+ \times R^n, R^+\right]$; $l \in C^1\left[R^+ \times R^n \times R^p, R^+\right]$. For simplicity of notation, we write G, l instead of $G(t,x), l(t,x,u)$, respectively.

Suppose that goal function J(u) is minimal at $u^*(t) \in U \subset R^p$, we have to find the necessary conditions for the SOCP .

Theorem 3. *With the goal function*

$$J(u) = \int_{H_0} G(T, x(T)) d\mu , \tag{8}$$

where $G \in C^1\left[R^+ \times R^n, R^+\right]$ *and* μ *-Lebesgue measure on* H_0, *the necessary conditions that* u^* *be an optimal control for the SOCP are*

$$\left[\frac{\partial G}{\partial x}\right]^T \left[\frac{\partial x}{\partial u^*}\right] = 0 \in R^p \text{ and}$$

$$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{\partial u^*}{\partial t}\right] = f(t, x, u^*) - \left[\frac{\partial x}{\partial t}\right] \tag{9}$$

$\forall\, t \in I, x_0 \in H_0$.

Proof. Suppose that u* is an optimal control for SOCP. Let $x(t,u^*)=x(t,x_0,u^*)$ be the solution of (1) where u* is a function of t, one has

$$\left[\frac{d}{dt}x(t,u^*)\right]=\left[\frac{\partial x}{\partial t}\right]+\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{\partial u^*}{\partial t}\right].$$

It implies (9) $$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{du^*}{dt}\right]=f(t,x,u^*)-\left[\frac{\partial x}{\partial t}\right].$$

Now, the goal function is considered as a function depends on only one variable u*. It is supposed that $u^*+\theta h\in U$ for some $\theta\in R$ (with $|\theta|$ is sufficiently small) for all $h\in U$ and use variational method

$$\frac{d}{d\theta}J(u^*+\theta h)\,|_{\theta=0}=\int_{H_0}\left[\frac{\partial G}{\partial x}\right]^T\left[\frac{\partial x}{\partial u^*}\right].h\,d\mu=0,\forall h\in U.$$

Hence, $\left[\frac{\partial G}{\partial x}\right]^T\left[\frac{\partial x}{\partial u^*}\right]=0\in R^p$. The proof completes.

Theorem 4. *With the goal function*

$$J(u)=\int_0^T\left(\int_{H_U}l(t,x,u)d\mu\right)dt, \tag{10}$$

where $l\in C^1\left[R^+\times R^n\times R^p,R^+\right]$ *and* μ *-Lebesgue measure on* $H_U=\bigcup_{u\in U,t\in I}H_{t,u}$, *necessary conditions that u* be an optimal control for the SOCP are:*

$$\left[\frac{\partial l}{\partial x}\right]^T\left[\frac{\partial x}{\partial u^*}\right]+\left[\frac{\partial l}{\partial u^*}\right]^T=0\in R^p \quad \text{and}$$

$$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{du^*}{dt}\right]=f(t,x,u^*)-\left[\frac{\partial x}{\partial t}\right] \tag{11}$$

$\forall\ t\in I,\ x_0\in H_0$.

The proof of Theorem 4 is analogue of that of theorem 3.

Theorem 5. *With the goal function*

$$J(u)=\int_0^T\left(\int_{H_{t,U}}l(t,x,u)d\mu\right)dt$$

where $l\in C^1\left[R^+\times R^n\times R^p,R^+\right]$ *and* μ *- Lebesgue measure on* $H_{t,U}=\bigcup_{u\in U}H_{t,u}$, *then the necessary conditions that u* be an optimal control for the SOCP are*

$$\left[\frac{\partial l}{\partial x}\right]^T\left[\frac{\partial x}{\partial u^*}\right]+\left[\frac{\partial l}{\partial u^*}\right]^T=0\in R^p \quad \text{and}$$

$$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{du^*}{dt}\right]=f(t,x,u^*)-\left[\frac{\partial x}{\partial t}\right]$$

$\forall\ t\in I,\ x_0\in H_0$.

The proof of Theorem 5 is analogue of that of theorem 3.

Theorem 6. *With the goal function*

$$J(u)=\int_0^T\left(\int_{H_U}l(t,x,u)d\mu\right)dt+\int_{H_U}G(T,x(T))\,d\mu, \tag{12}$$

where $G \in C^1\left[R^+ \times R^n, R^+\right]$; $l \in C^1\left[R^+ \times R^n \times R^p, R^+\right]$ *and* μ *-Lebesgue measure on* H_U, *then the necessary conditions that u* be an optimal control for the SOCP are:*

$$\left[\frac{\partial l}{\partial x}\right]^T \left[\frac{\partial x}{\partial u^*}\right] + \left[\frac{\partial l}{\partial u^*}\right]^T + \frac{1}{T}\left\{\left[\frac{\partial G}{\partial x}\right]^T \left[\frac{\partial x}{\partial u^*}\right]\right\} = 0 \in R^p \qquad (13)$$

$$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{du^*}{dt}\right] = f(t,x,u^*) - \left[\frac{\partial x}{\partial t}\right]$$

The proof of Theorem 6 is analogue of that of theorem 3.

Theorem 7. *With the goal function*

$$J(u) = \int_0^T \left(\int_{H_0} l(t,x,u)d\mu \right) dt + \int_{H_0} G(T, x(T))\, d\mu, \qquad (14)$$

where $G \in C^1\left[R^+ \times R^n, R^+\right]$; $l \in C^1\left[R^+ \times R^n \times R^p, R^+\right]$ *and* μ *-Lebesgue measure on* H_0, *then the necessary conditions that u* be an optimal control for the SOCP are*

$$\left[\frac{\partial l}{\partial x}\right]^T \left[\frac{\partial x}{\partial u^*}\right] + \left[\frac{\partial l}{\partial u^*}\right]^T + \frac{1}{T}\left\{\left[\frac{\partial G}{\partial x}\right]^T \left[\frac{\partial x}{\partial u^*}\right]\right\} = 0 \in R^p$$

$$\left[\frac{\partial x}{\partial u^*}\right]\left[\frac{du^*}{dt}\right] = f(t,x,u^*) - \left[\frac{\partial x}{\partial t}\right]$$

The proof of Theorem 7 is analogue of that of theorem 3.

Remark. Theorem 4 is a special case of Theorem 6 and Theorem 3 is a special case of Theorem 7. The SOCPs in theorems 6, 7 are called Bolza like problems, in theorem 3 is called Mayer like problem, in theorems 4, 5 are called Lagrange like problems.

5. CONCLUSION

In this paper concepts of SOCP, SOCPF and their dependences on given controls were presented. Some first results of necessary conditions for SOCP were shown.

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VỀ CÁC ĐIỀU KIỆN CẦN CHO BÀI TOÁN ĐIỀU KHIỂN TỐI ƯU BÓ DẠNG MỜ

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TÓM TẮT: *Một trong những thế mạnh của Lý thuyết mờ là có thể giải quyết những bài toán điều khiển phi tuyến phức tạp. Bên cạnh các dạng mờ có thể tìm các nghiệm điều khiển.*

Trong bài báo này, chúng tôi muốn giới thiệu bài toán điều khiển bó (SOCP), nghĩa là điều khiển hàng loạt quá trình và bài toán điều khiển tối ưu bó dạng mờ (SOCPF).

Từ khóa: *Lý thuyết mờ, Lý thuyết điều khiển toi uu, Phương trình vi phân.*

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