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Research article

Annihilators of top local cohomology modules and catenarity of rings

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ABSTRACT
Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R and M a finitely generated R-module with $\dim_R(M) = d$. The annihilator of the top local cohomology module $H_I^d(M)$ related to the catenarity of the ring is given in this paper. The main result implies that the ring R is catenary if and only if the annihilator of the top local cohomology modules is compatible under localization.
Key words: Top local cohomology module, annihilator, catenarity

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INTRODUCTION

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R and M a finitely generated R-module with $\dim_R(M) = d$. We denote by $\mathrm{Var}(I)$ the set of all prime ideals containing I. The i-th local cohomology $H_I^d(M)$ of M with respect to I is defined by $H_I^d(M) = \varinjlim_{n\in\mathbb{N}} Ext_I^i(R/I^n, M)$.

Annihilators of local cohomology modules play an important role in the study of some Homology Conjecture [1, Introduction]. Annihilators of local cohomology modules are also related to the structure of rings (see [2–4]). A formula for the top local cohomology module was given in [1,5–7]. Assume that $0 = \bigcap_{\mathfrak{p}\in Ass_R(M)} N(\mathfrak{p})$ is the reduced primary decomposition of 0. Set

$$U_M(0) := \bigcap_{\mathfrak{p}\in Ass_R(M), dim(R/\mathfrak{p})=d} N(\mathfrak{p})$$

By [7, Corollary 1.3] (see also [5, Theorem 2.6]),

$$Ann_R H_{\mathfrak{m}}^d(M) = Ann_R M/U_M(0). \quad (1)$$

Note that $Ann_{R_\mathfrak{p}} M_\mathfrak{p} = (Ann_R M)_\mathfrak{p}$ for every finitely generated R-module M and for every ideal $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Naturally, we ask whether there is an analog property for Artinian modules. It seems difficult to find a suitable notion of "co-localization" or "dual to localization" for Artinian modules with certain necessary properties (see [8]). So, we concentrate on Artinian local cohomology modules $H_{\mathfrak{m}}^i(M)$ with support in the maximal ideal $\mathfrak{m}$. By Local Duality Theorem (see [9]), $H_{\mathfrak{p}R_\mathfrak{p}}^{i-Rdim(R/\mathfrak{p})}(M_\mathfrak{p})$ is considered as "co-localization" of $H_{\mathfrak{m}}^i(M)$. So it is natural to consider the relation between $Ann_{R_\mathfrak{p}} H_{\mathfrak{p}R_\mathfrak{p}}^{i-Rdim(R/\mathfrak{p})}(M_\mathfrak{p})$ and $(Ann_R H_{\mathfrak{m}}^i(M))R_\mathfrak{p}$.

Recall that a ring is called catenary if for any two prime ideals $\mathfrak{p} \subset \mathfrak{p}'$, two maximal chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{p}'$ have the same length. Now let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension 3 such that R is not catenary (see [10, Appendix, Example 2]). Since R is not catenary and $\dim R = 3$, there exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} + \mathrm{ht}(\mathfrak{p}) = 2$. This follows that $\dim R/\mathfrak{p} = 1$ and $\mathrm{ht}(\mathfrak{p}) = 1$. By the formula (1) for annihilators of top local cohomology modules, we have

$$\left(Ann_R H_{\mathfrak{m}}^3(R)\right)R_\mathfrak{p} = (Ann_R(R/U_R(0)))R_\mathfrak{p} = Ann_R(R)R_\mathfrak{p} = 0_{R_\mathfrak{p}}$$

and $Ann_{R_\mathfrak{p}} H_{\mathfrak{p}R_\mathfrak{p}}^{3-dim(R/\mathfrak{p})}(R_\mathfrak{p}) = Ann_{R_\mathfrak{p}} 0 = R_\mathfrak{p}$. So

$$Ann_{R_\mathfrak{p}} H_{\mathfrak{p}R_\mathfrak{p}}^{3-dim(R/\mathfrak{p})}(R_\mathfrak{p}) \neq \left(Ann_{R_\mathfrak{p}} H_{\mathfrak{m}}^3(R)\right)R_\mathfrak{p}$$

In [2], the author and L. T. Nhan showed that R is catenary if and only if

$$Ann_{R_\mathfrak{p}} H_{\mathfrak{p}R_\mathfrak{p}}^{dim(M)-dim(R/\mathfrak{p})}(M_\mathfrak{p}) = \left(Ann_R H_{\mathfrak{m}}^{dim(M)}(M)\right)R_\mathfrak{p}$$

for every finitely generated R-module M and every $\mathfrak{p} \in \mathrm{Spec}(R)$. In this paper, we extend the above results to Artinian modules $H_I^d(M)$ for each given finitely generated R-module M. The main result is presented in the next section.

MAIN RESULT

Assume $0 = \bigcap_{\mathfrak{p}\in Ass_R(M)} N(\mathfrak{p})$ is the reduced primary decomposition of 0. The set of associated primes of highest dimension is denoted by $\mathrm{Assh}_R(M)$, i.e.

$$Assh_R(M) = \{\mathfrak{p} \in Ass_R(M) \mid dim(R/\mathfrak{p}) = d\}.$$

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Set

$$U_M(0) := \bigcap_{p \in Assh_R(M)} N(p).$$

Note that $U_M(0)$ depends only on 0 and M and it does not depend on the reduced primary decomposition of 0. Moreover, $U_M(0)$ is the largest submodule of M of dimension less than $\dim(M)$. The theory of *secondary representation* and the set of *attached prime ideals* which are dual to the theory of primary decomposition and associated prime ideals respectively were introduced by I. G. Macdonald ([11]). It is known that every Artinian R-module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is pi-secondary. The set $\{p_1, \ldots, p_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called the set of *attached prime ideals* of A, and denoted by $Att_R(A)$. Remind that local cohomology modules with maximal support $H^i_m(M)$ are Artinian (see [9, 7.1.3]). Attached prime ideals of top local cohomology modules $H^d_m(M)$ and associated prime ideals of M have the following relation (see [9, 7.3.2]).

Lemma 2.1. $Att_R(H^d_m(M)) = Assh_R(M)$

Then

$$U_M(0) = \bigcap_{p \in Att_R(H^d_m(M))} N(p).$$

The following result was proved in [7, Corollary 1.3] (see also [5, Theorem 2.6]).

Lemma 2.2. $Ann_R H^d_m(M) = Ann_R M/U_M(0)$.

It is known that the top local cohomology module $H^d_I(M)$ is Artinian (see [9, 7.1.6]). The attached prime ideals of $H^d_I(M)$ were given in [12]. Recall that for a finitely generated R-module L, the cohomological dimension of L with respect to an ideal I of R is defined as follows

$$cd(I, L) := sup\left\{ i \in \mathbb{Z} \mid H^i_I(L) \neq 0 \right\}$$

and $cd(m, L) = \dim_R L$ by the Non-Vanishing Theorem (see [9, 6.1.4])

Lemma 2.3. *([12, Theorem A])* $Att_R(H^d_I(M)) = \{p \in Ass_R(M) | cd(I, R/p) = d\}$.

Set $N* = \bigcap_{p \in Att_R(H^d_I(M))} N(p)$. Since $Att_R(H^d_I(M)) \subseteq Assh_R(M)$, $U_M(0) \subseteq N^*$. Furthermore if $I = m$ then $U_M(0) = N^*$. We have the following result on the annihilator of the top local cohomology module $H^d_I(M)$ (see [7]).

Lemma 2.4. *Assume that* $H^d_I(M) \neq 0$. *Then*

(i) $N^* = \bigcap_{cd(I,R/p)=d} N(p) = H^0_b(M)$ *where* $b = \prod_{cd(I,R/p)\neq d}$. *Further more,* N^* *is the largest submodule of M such that* $cd(I, N^*) < d$.

(ii) $Ann_R H^d_I(M) = Ann_R M / \bigcap_{cd(I,R/p)=d} N(p) = Ann_R M/N^*$.

Therefore, we naturally consider the relation between $Ann_{R_p} H^{d-pdim(R/p)}_{pR_p}((M/N^*)p)$ and $\left(Ann_R H^d_I(M)\right) R_p$ where $p \in Spec(R)$. The following theorem which is the main result of this paper clarifies the structure of local rings such that the above two annihilators are equal.

Theorem 2.5. *The following statements are equivalent:*

(i) $R/Ann_R H^d_I(M)$ *is catenary;*

(ii) $Ann_{R_p} H^{d-pdim(R/p)}_{pR_p}((M/N^*)p) = \left(Ann_R H^d_I(M)\right) R_p$ *for all* $p \in Spec(R)$.

Proof. (i) $\Rightarrow$ (ii). Let $p \in Spec(R)$. By Lemma 2.4, $Ann_R H^d_I(M) = Ann_R M/N^*$. Thus if $p \notin Var(Ann_R M/N^*)$, then $(M/N^*)_p = 0$ and $p \notin Var(Ann_R H^d_I(M))$. In this case we get that $Ann_{R_p} H^{d-pdim(R/p)}_{pR_p}((M/N^*)p) = \left(Ann_R H^d_I(M)\right) R_p = R_p$

Now we assume that $p \supseteq Ann_R M/N^*$. By the assumption (i), $R/Ann_R H^d_I(M)$ is catenary. Since $Ann_R M/N = Ann_R H^d_I(M)$, M/N^* is equidimentional, that is $\dim R/p = \dim M/N^*$ for all $p \in \min Ass_R M$. We get that

$$dim(M/N^*)_p = dim(M/N^*) - dim(R/p)$$
$$= d - dim(R/p).$$

Note that $N^* = \bigcap_{p \in Att_R(H^d_I(M))} N(p)$ is a primary decomposition of submodule N^* in M and $Att_R(H^d_I(M)) \subseteq Assh_R(M)$. Thus $dim(Rp/qR_p) = dim(M/N^*)_p$ for all $qRp \in Ass_{R_p}(M/N^*)_p$. Then $U_{(M/N^*)_p}(0) = 0$. By Lemma 2.2, $Ann_{R_p} H^{d-dim(R/p)}_{pR_p}((M/N^*)p) = Ann_{R_p}((M/N^*)p)/U_{(M/N^*)_p}(0) = Ann_{R_p}(M/N^*)_p$. By Lemma 2.1, $\left(Ann_R H^d_I(M)\right) R_p = Ann_{R_p}(M/N^*)_p$. So we get (ii).

(ii) $\Rightarrow$ (i). We can assume that $H^d_I(M) \neq 0$. Let $q \in min\left(Var(Ann_R H^d_I(M))\right)$. We need to prove that R/q is catenary. Let $p \in Spec(R)$ such that $p \supseteq q$. Then we have $qR_p \in min\left(Ann_R H^d_I(M)\right) Rp$. By (ii), $qR_p \in min\left(Ann_{R_p} H^{d-dim(R/p)}_{pR_p}((M/N^*)_p)\right)$. Hence by [11], $qR_p \in Att_R(H^{d-dim(R/p)}_{pR_p}((M/N^*)_p))$. This follows $H^{d-dim(R/p)}_{pR_p}((M/N^*)_p) \neq 0$. Since $H^{d-dim(R/p)}_{pR_p}((M/N^*)_p) \neq 0$, $dim(M/N^*)_p = d - dim(R/p)$. As

$$qR_p \in Att_R(H^{d-dim(R/p)}_{pR_p}((M/N^*)_p)),$$

by Lemma 2.1, we get that

$$dim(R_p/qR_p) = d - dim(R/p),$$

i.e. ht(p / q) = d − dim(R/ p). Since $q \in min\left(Var\left(Ann_R H_I^d(M)\right)\right)$, $q \in Att_R(H_I^d(M))$ and then dim(R/q) = d. Hence

$$ht(p/q) = dim(R/q) - dim(R/p).$$

This proves that R/q is catenary. By [13, Theorem 2.2], $R/Ann_R(H_I^d(M))$ is catenary. □

We have the following corollary that is one of the main results in[2].

Corollary 2.6. *The following statements are equivalent:*

(i) R is catenary.

(ii) $Ann_{R_p} H_{pR_p}^{dim(M)-dim(R/p)}(M_p) = \left(Ann_R H_m^{dim(M)}(M)\right) R_p$ *for every finitely generated R-module M, every* $p \in Spec(R)$ *and every integer* $i \geq 0$.

Proof. Let M be a finitely generated R-module. If I = m then $N^* = U_M(0)$. Since $\dim(U_M(0))_p < d - \dim(R/p)$, from the exact sequence

$$0 \to (U_M(0))_p \to M_p \to (M/U_M(0))_p \to 0,$$

we get that

$$H_{pR_p}^{d-dim(R/p)}\left((M/U_M(0))_p\right) \cong H_{pR_p}^{d-dim(R/p)}\left((M)_p\right).$$

(i) ⇒ (ii) follows from Theorem 2.5 and (1).

(ii) ⇒ (i). Let p ∈ Spec(R). Note that $Ann_R\left(H_m^{dim(R/p)}(R/p)\right) = p$. Set M = R/p. By (ii), Theorem 2.5 and (1), we get that R/p is catenary. So, R is catenary. □

COMPETING INTERESTS

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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REFERENCES

1. Lynch LR. Annihilators of local cohomology. Dissertation for the degree of doctor of philosophy;Available from: https://digitalcommons.unl.edu/mathstudent/21.
2. Hang NA, Nhan LT. Annihilator of local cohomology modules under localization and completion. submitted;.
3. Nhan LT, An TN. On the unmixedness and the universal catenaricity of local rings and local cohomology modules. J. Algebra. 2009;321:303-11;Available from: https://doi.org/10.1016/j.jalgebra.2008.09.005.
4. Nhan LT, Chau TM. On the top local cohomology modules. J. Algebra. 2012;349:342-52;Available from: https://doi.org/10.1016/j.jalgebra.2011.08.027.
5. Bahmanpour K, A'zami J, Ghasemi G. On the annihilators of local cohomology modules. J. Algebra. 2012;363:8-13;Available from: https://www.sciencedirect.com/science/article/pii/S0021869312001780.
6. Lynch LR. Annihilators of top local cohomology. Comm. Algebra. 2012;40:542-51;Available from: https://doi.org/10.1080/00927872.2010.533223.
7. Atazadeh A, Sedghi M, Naghipour R. Cohomological dimension filtration and annihilators of top local cohomology modules. Coll. Math. 2015;139:25-35;Available from: https://eudml.org/doc/284334.
8. Melkersson L, Schenzel P, The co-localization of an Artinian module. Proc. Edinb. Math. Soc. 1995;38,121-31;Available from: https://doi.org/10.1017/S0013091500006258.
9. Brodmann M, Sharp RY. Local cohomology: an algebraic introduction with geometric applications. Cambridge University Press; 1998;Available from: https://doi.org/10.1017/CBO9781139044059.
10. Nagata M. Local rings. New York: Interscience; 1962;.
11. Macdonald IG, Secondary representation of modules over a commutative ring. Symposia Math. 1973;11:23-43;.
12. Dibaei TM, Yassemi S, Attached primes of the top local cohomology modules with respect to an ideal. Arch. Math. 2005;84:292-97;Available from: https://doi.org/10.1007/s00013-004-1156-2.
13. Ratliff LJ. On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals (I). Amer. J. Math. 1969;91:508-28;Available from: https://doi.org/10.2307/2373524.