Sequential Farkas lemmas for convex systems

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ABSTRACT

In this paper we introduce two new versions of Farkas lemma for two kinds of convex systems in locally convex Hausdorff topological vector spaces which hold without any constraint qualification conditions. These versions hold in the limits and will be called sequential Farkas lemmas. Concretely, we establish sequential Farkas lemmas for cone-convex systems and for systems which are convex with respect to a sublinear function. The first result extends some known ones in the literature while the second is a new one.

Keywords: *The Farkas lemma, sequential Farkas lemmas, functional inequalities, approximate Hahn-Banach theorem.*

INTRODUCTION

Farkas lemma is one of the most important results from fundamental mathematics. It is equivalent to the Hahn-Banach theorem [10] and has had many applications in economics [9], in finance [8], in mechanics, and in many fields of applied mathematics such asmathematical programming and optimal control. For more details, see the recent survey paper [7].

The first correct version of Farkas lemma for a linear system was introduced by the physicist Gyula Farkas in 1902. Since then, many generalized versions of this “lemma" have been proved. Most of these extensions are non-asymptotic versions and hold under some qualification conditions [5, 7]. In the recent years, several asymptotic versions for generalized Farkas lemma have been established and found many applications in optimization problems [4, 6, 11].

In this paper, we first introduce an asymptotic version of Farkas lemma for a cone-convex system which extends some known results in the literature [6, 11]. Imitating the idea in [5], we then establish the corresponding asymptotic version of Farkas lemma for systems which are convex with respect to a sublinear function, which appears for the first time and may pay the way for applications to some areas in applied mathematics.

NOTATIONS AND PRELIMINARIES

Let X and Y be locally convex Hausdorff topological vector spaces (l.c.H.t.v.s.), with their topological dual spaces X^* and Y^*, endowed with w^*-topologies, respectively. Given a set $A \subset X^*$, we denote by $\text{cl}A$ the closure of A (with respect to the w^*-topology) in X^*.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$. The effective domain of f is $\text{dom} f := \{x \in X : f(x) < +\infty\}$. The function f is said to be proper if $\text{dom} f \neq \emptyset$. The epigraph of f is $\text{epi} f := \{(x,\alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$. The set of all proper, lower semi-continuous (lsc) and convex functions on X is denoted by $\Gamma(X)$.

The conjugate function regarding the set $C \subset X$ of f is the function $f_C^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f_C^*(x^*) = \sup_{x \in C} \{\langle x^*, x\rangle - f(x)\}, \forall x^* \in X^*.$$

When $C = X$ the conjugate regarding the set C is the classical (Fenchel-Moreau) conjugate function of f denoted by f^* see [3].

The indicator function of the set $A \subset X$ is denoted by i_A, *i.e.,* $i_A(x) = 0$ if $x \in A$, $i_A(x) = +\infty$ if $x \in X \setminus A$.

A closed convex cone $K \subset Y$ generates a partial order $\leq_K$ on Y by

$$y_1 \leq_K y_2 \text{ if } y_2 - y_1 \in K.$$

We add to Y a greatest element with respect to $\leq_K$, denoted by ∞_K, which does not belong to Y and let $Y^\bullet = Y \cup \{\infty_K\}$. Then one has $y \leq_K \infty_K$ for every $y \in Y^\bullet$. We shall use the following conventions on $Y^\bullet$: $y + \infty_K = \infty_K + y = \infty_K$, for all $y \in Y^\bullet$, and $\alpha\infty_K = \infty_K$ if $\alpha \geq 0$. The dual cone of K, denoted by K^+, is defined by

$$K^+ := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in K\}.$$

A mapping $h : X \to Y^\bullet$ is called (extended) K-convex if

$$x_1, x_2 \in X,\ \mu_1, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$$
$$\Rightarrow h(\mu_1 x_1 + \mu_2 x_2) \leq_K \mu_1 h(x_1) + \mu_2 h(x_2),$$

where "$\leq_K$" is the binary relation (generated by K) extended to $Y^\bullet$. The domain of h, denoted by $\text{dom}\, h$, is defined as the set $\text{dom}\, h := \{x \in X : h(x) \in Y\}$ and the $K-$epigraph of h is:

$$\text{epi}_K h := \{(x, y) \in X \times Y : y \in h(x) + K\}.$$

It is clear that h is K-convex if and only if $\text{epi}_K h$ is convex. The mapping $h : X \to Y^\bullet$ is said to be $K-$ epi-closed if $\text{epi}_K h$ is a closed subset in the product space $X \times Y$, and in this case, the set $h^{-1}(-K)$ is closed as $h^{-1}(-K) \times \{0_Y\} = (\text{epi}_K h) \cap (X \times \{0_Y\})$ [5]. It is also worth observing that if K is a closed convex cone and h is a K-convex function then $h^{-1}(-K)$ is a convex subset of X. Moreover, for any $y^* \in Y^*$ the function $y^* \circ h$ is defined on X as follows:

$$(y^* \circ h)(x) = \begin{cases} \langle y^*, h(x)\rangle, & \text{if } x \in \text{dom}\, h, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $S : Y \to \mathbb{R} \cup \{+\infty\}$ is called (extended) sublinear if

$$S(y + y') \leq S(y) + S(y'), \text{ and } S(\lambda y) = \lambda S(y), \forall y, y' \in Y, \forall \lambda > 0.$$

By convention, we set $S(0_Y) = 0$. Such a function S can be extended to $Y^\bullet$ by setting $S(\infty_K) = +\infty$ together with all other conventions related to the operations in $Y^\bullet$ defined above.

An extended sublinear function $S: Y \to \mathbb{R} \cup \{+\infty\}$ allows us to introduce in $Y^\bullet$ a binary relation which is reflexive and transitive:

$$y_1 \leq_S y_2 \text{ if } y_1 \leq_K y_2,$$
$$\text{where } K := \{y \in Y : S(-y) \leq 0\}.$$

Here the relation "$\leq_K$" is used in the extended sense to $Y^\bullet$.

Given an extended sublinear function $S: Y \to \mathbb{R} \cup \{+\infty\}$, we adapt the notion S-convex (*i.e.*, convex with respect to a sublinear function) (see [13]) and introduce the one corresponding to an extended sublinear function S:

A mapping $h: X \to Y^\bullet$ is said to be (extended) S-convex if for all $x_1, x_2 \in X$, $\mu_1, \mu_2 > 0$, $\mu_1 + \mu_2 = 1$, one has

$$h(\mu_1 x_1 + \mu_2 x_2) \leq_S \mu_1 h(x_1) + \mu_2 h(x_2).$$

It can be verified easily that if h is S-convex then h is K-convex with $K := \{y \in Y : S(-y) \leq 0\}$. Conversely, if h is K-convex with some convex cone K then h is S-convex with $S = i_{-K}$.

Some properties of limit inferior and limit superior of nets of extended real numbers will be quoted in the next lemma.

Lemma 1 [1, 2] *Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be nets of extended real numbers. Then the following statements hold:*

(i) $\liminf_{i\in I} (a_i + b_i) \geq \liminf_{i\in I} a_i + \liminf_{i\in I} b_i$ and $\limsup_{i\in I} (a_i + b_i) \leq \limsup_{i\in I} a_i + \limsup_{i\in I} b_i$, *provided that the right hand side of the inequalities are well-defined. Moreover, the equalities hold whenever one of the nets is convergent.*

(ii) *If* $s \geq 0$ *then* $\liminf_{i\in I} (sa_i) = s \liminf_{i\in I} a_i$ *and* $\limsup_{i\in I} (sa_i) = s \limsup_{i\in I} a_i$; *if* $s \leq 0$ *then* $\liminf_{i\in I} (sa_i) = s \limsup_{i\in I} a_i$ *and* $\limsup_{i\in I} (sa_i) = s \liminf_{i\in I} a_i$.

A net $(u_i^*)_{i\in I}$ in the topological dual space X^* of an l.c.H.t.v.s. X converges to $u^* \in X^*$ w.r.t. the w^*-topology will be denoted by $u_i^* \to^* u^*$.

SEQUENTIAL FARKAS LEMMA FOR K-CONVEX SYSTEMS

In this section we will introduce a version of Farkas lemma for cone-convex systems which holds in the limit form without any qualification condition. The result extends the ones in [6], [11] and [4] in some senses.

Theorem 2 [Approximate Farkas lemma 1] *Let X, Y be locally convex Hausdorff topological vector spaces, K be a closed convex cone in Y, C be a nonempty closed convex subset of X, $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lsc function. Let further, $g: X \to Y^\bullet$ be a $K-$convex and K-epi closed mapping and $\beta \in \mathbb{R}$. Assume that $(dom f) \cap C \cap g^{-1}(-K) \neq \emptyset$. Then the following statements are equivalent:*

(i) $x \in C, g(x) \in -K \Rightarrow f(x) \geq \beta.$

(ii) *There exists a net $(y_i^*)_{i\in I} \subset K^+$ such that*

$$f(x) + \liminf_{i\in I} (y_i^* \circ g)(x) \geq \beta, \forall x \in C. \quad (1)$$

Proof. • [(i) $\Rightarrow$ (ii)] Assume that (i) holds. Let $A := C \cap g^{-1}(-K)$. It is obvious that A is non-empty closed convex set (as C and $g^{-1}(-K)$ are closed and convex). Observe that (i) is equivalent to $-\beta \geq (f + i_A)^*(0_{X^*})$, or equivalently, $(0_{X^*}, -\beta) \in \text{epi}(f + i_A)^*$. By using Remark 8.11, p.54 in [2] one has $\text{epi}(f + i_A)^* = \text{cl}(\text{epi}\, f^* + \bigcup_{\lambda \in K^+} \text{epi}(\lambda g + i_C)^*)$.

Consequently, (i) is equivalent to $(0_{X^*}, -\beta) \in \text{cl}(\text{epi}\, f^* + \bigcup_{\lambda \in K^+} \text{epi}(\lambda g + i_C)^*)$.

Thus, there are nets $(x_i^*)_{i\in I}, (v_i^*)_{i\in I}, \subset X^*$, $(y_i^*)_{i\in I} \subset K^+$, $(\alpha_i)_{i\in I} \subset \mathbb{R}_+$, such that

$$x_i^* + v_i^* \to^* 0_{X^*},$$

$$\lim_{i\in I} [f^*(x_i^*) + (y_i^* \circ g + i_C)^*(v_i^*) + \alpha_i] = -\beta.$$

For each $i \in I$, set $\varepsilon_i := f^*(x_i^*) + (y_i^* \circ g + i_C)^*(v_i^*) + \alpha_i + \beta$. Then $\varepsilon_i \to 0$. Note also that for any $x \in C$ one has

$$f^*(x_i^*) \geq \langle x_i^*, x\rangle - f(x)$$

and $(y_i^* \circ g + i_C)^*(v_i^*) \geq \langle v_i^*, x\rangle - (y_i^* \circ g)(x)$, and so, it follows from the definition of ε_i and $\alpha_i \geq 0$, one gets:

$$\varepsilon_i \geq \langle x_i^* + v_i^*, x\rangle - f(x) - (y_i^* \circ g)(x) + \beta, \quad \forall x \in C, \forall i \in I. \tag{2}$$

Taking the limit superior both sides of (2) we get

$$0 = \limsup_{i\in I} \varepsilon_i \geq -f(x) + \beta + \limsup_{i\in I} (\langle x_i^* + v_i^*, x\rangle - (y_i^* \circ g)(x)), \forall x \in C,$$

or equivalently,

$$f(x) - \beta \geq \limsup_{i\in I} (\langle x_i^* + v_i^*, x\rangle - (y_i^* \circ g)(x)), \forall x \in C \tag{3}$$

Since $x_i^* + v_i^* \to^* 0_{X^*}$, it follows from (3) that, for all $x \in C$,

$$f(x) - \beta \geq \limsup_{i\in I} (\langle x_i^* + v_i^*, x\rangle - (y_i^* \circ g)(x)) + \limsup_{i\in I} (-\langle x_i^* + v_i^*, x\rangle)$$
$$\geq \limsup_{i\in I} (-(y_i^* \circ g)(x)) = -\liminf_{i\in I} (y_i^* \circ g)(x)$$

(see Lemma 1). Consequently, we arrive at (ii):

$$f(x) + \liminf_{i\in I} (y_i^* \circ g)(x) \geq \beta, \quad \forall x \in C.$$

- [(ii) $\Rightarrow$ (i)] Assume that (ii) holds, i.e., there exists a net $(y_i^*)_{i\in I} \subset K^+$ such that

$$f(x) + \liminf_{i\in I} (y_i^* \circ g)(x) \geq \beta, \forall x \in C.$$

Observe that if $x \in C$ such that $g(x) \in -K$, then $(y_i^* \circ g)(x) \leq 0$ for all $i \in I$. Therefore, for $x \in C$ with $g(x) \in -K$ one gets

$$f(x) \geq f(x) + \liminf_{i\in I} (y_i^* \circ g)(x) \geq \beta.$$

The proof is complete.

Remark 3 *The equivalence between (i) and (ii) in Theorem 2 was established in [6], [11] under the assumption that $C = X$ and g is a continuous, K-convex mapping with values in Y. This equivalence was also established in [4] by another approach, called dual approach, for the case where $g : X \to Y$ is a K-convex mapping and satisfies the assumption that $y^* \circ g$ is lsc for each $y^* \in K^+$, which is much stronger than our assumption that g is K-epi-closed (see [2]). Our result extends all the results in the mentioned papers.*

SEQUENTIAL FARKAS LEMMA FOR SUBLINEAR-CONVEX SYSTEMS

In this section, we will establish a sequential version of Farkas lemma for systems of inequalities involving sublinear-convex mappings. The key tools used here are the technique of switching a sublinear-convex system to a cone-convex system used in the recent paper [5] and Theorem 2 from the previous section.

Theorem 4 [Approximate Farkas lemma 2]

Let X, Y be l.c.H.t.v.s., C be a nonempty closed convex subset of X, $S : Y \to \mathbb{R} \cup \{+\infty\}$ be an extended lsc sublinear function, $g : X \to Y^\bullet$ be an S-convex mapping such that the set

$$\{(x, y, \lambda) \in X \times Y \times \mathbb{R} : S(g(x) - y) \le \lambda\} \quad (4)$$

is closed in the product space $X \times Y \times \mathbb{R}$. Let us consider two proper convex and lsc functions $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. Assume that

$$(\mathrm{dom} f) \cap \{x \in C : \exists \alpha \in (\mathrm{dom}\psi) \cap \mathbb{R}_+ \text{ s.t. } (S \circ g)(x) < \alpha\} \neq \emptyset \quad (5)$$ [1]

Then the following statements are equivalent:

(*a*) $x \in C, \alpha \in \mathbb{R}_+, (S \circ g)(x) \le \alpha \Rightarrow f(x) + \psi(\alpha) \ge 0.$

(*b*) *There exist nets $(\gamma_i)_{i \in I} \subset \mathbb{R}_+$ and $(y_i^*)_{i \in I} \subset Y^*$ such that $y_i^* \le \gamma_i S$ for all $i \in I$ and*

$$f(x) + \liminf_{i \in I} (y_i^* \circ g)(x) \ge \psi^*_{\mathbb{R}_+}(\liminf_{i \in I} \gamma_i), \forall x \in C. \quad (6)$$

Proof. $[(a) \Rightarrow (b)]$ Assume (a) holds. Let us set $\tilde{Y} = Y \times \mathbb{R}$, $\tilde{Y}^\bullet = Y^\bullet \times \mathbb{R}$, $\tilde{C} = C \times \mathbb{R}_+$ and set $\tilde{S} : \tilde{Y} \to \mathbb{R} \cup \{+\infty\}$ defined by $\tilde{S}(y, \lambda) = S(y) - \lambda$ for all $(y, \lambda) \in \tilde{Y}$. Then $\tilde{C}$ is a nonempty closed convex subset of $X \times \mathbb{R}$, $\tilde{S}$ is an lsc sublinear function. Let also $\tilde{X} := X \times \mathbb{R}$, $\tilde{g} : \tilde{X} \to \tilde{Y}^\bullet$ and $\tilde{f} : \tilde{X} \to \mathbb{R} \cup \{+\infty\}$ be the mappings defined by

$$\tilde{g}(x, \alpha) := (g(x), \alpha), \forall (x, \alpha) \in \tilde{X}$$

and $\tilde{f}(x, \alpha) = f(x) + \psi(\alpha), \forall (x, \alpha) \in \tilde{X}.$

Since f, ψ are proper convex, lsc functions, so is $\tilde{f}$. Also, $\tilde{g}$ is an $\tilde{S}$-convex mapping as g is an S-convex mapping.

Now let $\tilde{K}$ be the closed convex cone defined by $\tilde{K} := \{(y, \lambda) \in \tilde{Y} : \tilde{S}(-y, -\lambda) \le 0\}$. Then it is easy to see that $\tilde{g}$ is $\tilde{K}$-convex as well.

The assumption (4) ensures $\tilde{g}$ is $\tilde{K}$–epi closed while (5) ensures that $(\mathrm{dom}\, \tilde{f}) \cap \tilde{C} \cap \tilde{g}^{-1}(-\tilde{K}) \neq \emptyset.$

We now apply Theorem 2 with $\tilde{X}$, $\tilde{Y}$, $\tilde{C}$, $\tilde{g}$, $\tilde{f}$, and $\tilde{K}$ playing the roles of X, Y, C, g, f, and K, respectively, and with $\beta = 0$.

From (*a*) and the definitions of $\tilde{f}$, $\tilde{g}$, we have

$$(x, \alpha) \in \tilde{C}, \tilde{g}(x, \alpha) = (g(x), \alpha) \in -\tilde{K} \Rightarrow \tilde{f}(x, \alpha) \ge 0,$$

which shows that (i) from Theorem 2 holds, and hence, by this theorem there exists a net $(\tilde{y}_i^*)_{i \in I} \subset \tilde{K}^+$ such that

$$\tilde{f}(x, \alpha) + \liminf_{i \in I} (\tilde{y}_i^* \circ \tilde{g})(x, \alpha) \ge 0, \forall (x, \alpha) \in \tilde{C}. \quad (7)$$

Since $(\tilde{y}_i^*)_{i \in I} \subset \tilde{K}^+$, by Lemma 3.5 in [5], there exists a net $(y_i^*, -\gamma_i)_{i \in I} \subset Y^* \times \mathbb{R}$ such that $\tilde{y}_i^* = (y_i^*, -\gamma_i)$, $\gamma_i \ge 0$ for each $i \in I$, and

$$y_i^* \le \gamma_i S \text{ for all } i \in I. \quad (8)$$

Therefore, (7) can be rewritten as

$$f(x) + \psi(\alpha) + \liminf_{i \in I} ((y_i^* \circ g)(x) - \gamma_i \alpha) \ge 0, \forall (x, \alpha) \in \tilde{C}. \quad (9)$$

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[1] When this condition holds, it is also said that the function is (S, g)-compatible [13]

It now follows from Lemma 5 below that $\liminf_{i\in I}\gamma_i \in \mathrm{P}_+$, and hence, for all $\alpha \in \mathrm{P}_+$, $\liminf_{i\in I}(\gamma_i\alpha)\in \mathrm{P}$. We get from Lemma 1 that

$$\liminf_{i\in I}(y_i^*\circ g)(x) - \liminf_{i\in I}(\gamma_i\alpha) \geq \liminf_{i\in I}((y_i^*\circ g)(x) - \gamma_i\alpha), \forall (x,\alpha)\in\tilde{C}.$$

Combining this inequality and (9) one gets, for all $(x,\alpha)\in\tilde{C}$,

$$0 \leq f(x) + \psi(\alpha) + \liminf_{i\in I}((y_i^*\circ g)(x) - \gamma_i\alpha)$$

$$\leq f(x) + \psi(\alpha) + \liminf_{i\in I}(y_i^*\circ g)(x) - \liminf_{i\in I}(\gamma_i\alpha),$$

which yields

$$f(x) + \liminf_{i\in I}(y_i^*\circ g)(x) \geq \liminf_{i\in I}(\gamma_i\alpha) - \psi(\alpha), \forall (x,\alpha)\in\tilde{C}.$$

Equivalently,

$$f(x) + \liminf_{i\in I}(y_i^*\circ g)(x) \geq (\liminf_{i\in I}\gamma_i)\alpha - \psi(\alpha), \quad \forall x\in C, \forall \alpha \in \mathbb{R}_+.$$

Taking the supremum over all $\alpha\in\mathrm{P}_+$ in the last inequality, we obtain

$$f(x) + \liminf_{i\in I}(y_i^*\circ g)(x) \geq \psi^*_{\mathbb{R}_+}(\liminf_{i\in I}\gamma_i), \ \forall x\in C,$$

which is (6).

$[(b)\Rightarrow(a)]$ Assume that (b) holds. Then there are two nets $(\gamma_i)_{i\in I}\subset \mathrm{P}_+$ and $(y_i^*)_{i\in I}\subset Y^*$ such that $y_i^*\leq \gamma_i S$ for all $i\in I$ and

$$f(x) + \liminf_{i\in I}(y_i^*\circ g)(x) \geq \liminf_{i\in I}(\gamma_i\alpha) - \psi(\alpha), \forall (x,\alpha)\in\tilde{C} \qquad (10)$$

Now if $x\in C, \alpha\in \mathrm{P}_+$ satisfy $(S\circ g)(x)\leq\alpha$ then $(y_i^*\circ g)(x)\leq \gamma_i(S\circ g)(x)\leq\gamma_i\alpha$ for all $i\in I$ and hence,

$$\liminf_{i\in I}(\gamma_i\alpha) \geq \liminf_{i\in I}(y_i^*\circ g)(x).$$

It follows from this inequality and (10) that for $x\in C, \alpha\in\mathrm{P}_+$ such that $(S\circ g)(x)\leq\alpha$, one gets

$$f(x) + \liminf_{i\in I}(\gamma_i\alpha) \geq \liminf_{i\in I}(\gamma_i\alpha) - \psi(\alpha), \qquad (11)$$

Since $\liminf_{i\in I}\gamma_i\in\mathrm{P}$ (see (6)), $\liminf_{i\in I}(\gamma_i\alpha)\in\mathrm{P}$ for any $\alpha\in\mathrm{P}_+$. Hence, (a) follows from (11). The proof is complete.

Lemma 5

With the notations used in the proof of Theorem 4, let $(y_i^*, -\gamma_i)_{i\in I}\subset Y^*\times\mathrm{P}$ *be such that* $\tilde{y}_i^* = (y_i^*, -\gamma_i)$, $\gamma_i\geq 0$ *for each* $i\in I$, *and* $y_i^*\leq\gamma_i S$ *for all* $i\in I$. *If*

$$f(x) + \psi(\alpha) + \liminf_{i\in I}((y_i^*\circ g)(x) - \gamma_i\alpha) \geq 0, \ \forall (x,\alpha)\in\tilde{C} \qquad (12)$$

then $\liminf_{i\in I}\gamma_i\in\mathrm{P}_+$.

Proof. Observe firstly that as $\gamma_i\geq 0$ for all $i\in I$ we have $\limsup_{i\in I}\gamma_i\geq\liminf_{i\in I}\gamma_i\geq 0.$ Let $\bar{x}\in C$ and $\bar{\alpha}\in\operatorname{dom}\psi$ be the elements that satisfy (5), i.e., $\bar{x}\in C\cap(\operatorname{dom} f)\cap(\operatorname{dom} g)$ and $\bar{\alpha}\in(\operatorname{dom}\psi)\cap\mathbb{R}_+$ such that

$$(S\circ g)(\bar{x}) < \bar{\alpha}, \quad (13)$$

and hence, $f(\bar{x})+\psi(\bar{\alpha}) < +\infty$. Since $y_i^*\leq\gamma_i S$ for each $i\in I$, one gets

$$(y_i^*\circ g)(\bar{x}) - \gamma_i\bar{\alpha} \leq \gamma_i(S\circ g)(\bar{x}) - \gamma_i\bar{\alpha} \leq \gamma_i((S\circ g)(\bar{x}) - \bar{\alpha}) \leq 0, \ \forall i\in I,$$

and hence,

$$\liminf_{i\in I}((y_i^*\circ g)(\bar{x}) - \gamma_i\bar{\alpha}) \leq \liminf_{i\in I}(\gamma_i(S\circ g)(\bar{x}) - \gamma_i\bar{\alpha}) \leq 0. \qquad (14)$$

Now, if (12) holds then one has

$$f(\bar{x}) + \psi(\bar{\alpha}) + \liminf_{i\in I}(\gamma_i(S\circ g)(\bar{x}) - \gamma_i\bar{\alpha}) \geq$$

$$f(\bar{x}) + \psi(\bar{\alpha}) + \liminf_{i\in I}((y_i^*\circ g)(\bar{x}) - \gamma_i\bar{\alpha}) \geq 0, \qquad (15)$$

which leads to

$f(\bar{x}) + \psi(\bar{\alpha}) \geq -\liminf_{i\in I}(\gamma_i((S\circ g)(\bar{x}) - \bar{\alpha}))$
$= (\limsup_{i\in I}\gamma_i)(\bar{\alpha} - (S\circ g)(\bar{x}))$, and hence (see also (13)),

$$\limsup_{i\in I}\gamma_i \leq \frac{f(\bar{x})+\psi(\bar{\alpha})}{\bar{\alpha}-(S\circ g)(\bar{x})} < +\infty.$$

Since $0\leq\liminf_{i\in I}\gamma_i\leq\limsup_{i\in I}\gamma_i$, one gets $\liminf_{i\in I}\gamma_i\in\mathrm{P}_+$.

Bổ đề Farkas theo dãy cho các hệ lồi

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TÓM TẮT

Trong bài báo này chúng tôi thiết lập hai phiên bản của Bổ đề Farkas mở rộng cho các hệ bất đẳng thức lồi trong không gian lồi địa phương. Các phiên bản này thoả mãn không cần bất kỳ điều kiện chính quy nào và chúng được thoả mãn dưới dạng giới hạn nên được gọi là các bổ đề Farkas suy rộng theo dãy. Cụ thể, chúng tôi thiết lập phiên bản Bổ đề Farkas suy rộng theo dãy cho hệ thống lồi theo nón và phiên bản cho hệ thống gồm các ánh xạ lồi theo một hàm dưới tuyến tính suy rộng cho trước. Phiên bản thứ nhất mở rộng nhiều kết quả của các tác giả khác trong khi phiên bản thứ hai là kết quả mới.

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