EXISTENCE OF SOLUTIONS OF SET CONTROL DIFFERENTIAL EQUATIONS

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ABSTRACT: *Recently, the field of differential equations has been studying in a very abstract method. Instead of considering the behaviour of one solution of a differential equation, one studies its sheaf-solution (see[8]). Instead of studying a differential equation, one studies differential inclusion (see[10]). Specially, one studies set differential equation (a differential equation whose variables and derivative are sets,(see[1-7]), a fuzzy differential equation is generalized to be* ***fuzzy control differential equation (FCDE),*** *(see[13]).*

In this paper, a set differential equation is generalized to be ***set control differential equation (SCDE)*** *and we present the existence and comparison of solutions of (SCDE).This paper is a continuation of our works in this direction (see [11-15]).*

Keywords: *Differential equations; Set differential equations; Set control differential equations.*

1. INTRODUCTION

In [1-8], authors studied the existence and comparison of solutions of the set differential equation (SDE)

$$D_H X(t) = F(t, X(t)),$$

where $X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), t \in [t_0, T] = I \subset R_+$

and $F : I \times K_c(R^n) \to K_c(R^n)$.

In this paper, we give the so-called set control differential equation (SCDE) in the form

$$D_H X(t) = F(t, X(t), U(t)),$$

where $X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), U(t) \in K_c(R^p),\ t \in [t_0, T] = I \subset R_+$

$F : I \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$,

and study existence of solutions and comparison solutions of SCDE.

The paper is organized as follows: we recall some basic concepts and notations which are useful in next sections in section 2 and some results on SDE in section 3. Existence results on solutions and comparison of solutions of SCDE are presented in section 4.

2. PRELIMINARIES

We recall some notations and concepts presented in detail in recent series works of V. Lakshmikantham et al. See [1-8].

Let $K_C(R^n)$ denote the collection of all nonempty, compact and convex subsets of R^n. Given A, B in $K_C(R^n)$. The Hausdorff distance between A and B defined as

$$D[A,B] = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \tag{2.1}$$

where $\|.\|$ denotes the Euclidean norm in R^n.

It is known that $\left(K_C(R^n), D\right)$ is a complete metric space and if the space $K_C(R^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_C(R^n)$ becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. See [15].

The Hausdorff metric satisfies some below properties.

$$D\left[A+C, B+C\right] = D[A,B] \text{ and } D[A,B] = D[B,A], \tag{2.2}$$

$$D\left[\lambda A, \lambda B\right] = \lambda D[B,A], \tag{2.3}$$

$$D[A,B] \le D\left[A,C\right] + D\left[C,B\right], \tag{2.4}$$

$$D[A+A', B+B'] \le D[A,B] + D\left[A', B'\right] \tag{2.5}$$

for all $A, B, C \in K_c(R^n)$ and $\lambda \in R_+$

Let $A, B \in K_c(R^n)$. The set $C \in K_c(R^n)$ satisfying $A = B + C$ is known as the geometric difference of the sets A and B and is denoted by the symbol $A - B$. Given an interval $I = \left[t_0, T\right] \subset R_+$. Function $F : I \to K_c(R^n)$ is said to be continuous at $t^* \in I$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t^*) > 0$ such that

$$D\left[F(t), F(t^*)\right] < \varepsilon \quad \text{for all} \quad t \in I \quad \text{with} \quad |t - t^*| < \delta.$$

We say that the mapping F has a Hukuhara derivative $D_H F(\tau)$ at a point $\tau \in I$, if

$$\lim_{h\to 0+} \frac{F(\tau+h) - F(\tau)}{h} \quad \text{and} \quad \lim_{h\to 0+} \frac{F(\tau) - F(\tau-h)}{h}$$

exist in the topology of $K_C(R^n)$ and are equal to $D_H F(\tau)$.

The Hukuhara integral of F is given by

$$\int_I F(s)ds = \left\{ \int_I f(s)ds : \ f \text{ is a continuous selector of } F \right\}$$

for any compact set $I \subset R_+$.

We have the following properties of the Hukuhara integral.

If $F : I \to K_c(R^n)$ is integrable, one has

$$\int_{t_0}^{t_2} F(s)ds = \int_{t_0}^{t_1} F(s)ds + \int_{t_1}^{t_2} F(s)ds,\ t_0 \le t_1 \le t_2 \tag{2.6}$$

and

$$\int_{t_0}^{t} \lambda F(s)ds = \lambda \int_{t_0}^{t} F(s)ds, \lambda \in R.$$

If $F, G : I \to K_c(R^n)$ are integrable, then $D[F(.), G(.)] : I \to R$ is integrable and

$$D\left[\int_{t_0}^{t} F(s)ds, \int_{t_0}^{t} G(s)ds\right] \le \int_{t_0}^{t} D\left[F(s), G(s)\right]ds. \tag{2.7}$$

Let us denote $D\left[A, \theta^n\right] = \|A\| = sup\{\|a\| : a \in A\}$

for $A \in K_c(R^n)$, where θ^n is the zero element of R^n which is regarded as a one point set. More details in continuity, Hukuhara derivative, Hukuhara integral of the mapping $F : I \to K_c(R^n)$, please see [1-8].

3. SET DIFFERENTIAL EQUATION

In [1-8], authors considered the set differential equation (SDE) as following.

$$D_H X(t) = F(t, X(t)), \tag{3.1}$$

where

$X(t_0) = X_0 \in K_c(R^n), X(t) \in K_c(R^n), t \in [t_0, T] = I$, state $X(t) \in K_c(R^n)$

and $F : I \times K_c(R^n) \to K_c(R^n)$.

The mapping $X \in C^1\left[I, K_c(R^p)\right]$ is said to be a solution of (3.1) on I if it satisfies (3.1) on I. Since $X(t)$ is continuously differentiable, we have

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s)ds,\ t \in I.$$

We associate with the initial value problem (3.1) the following

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s))ds,\ t \in I \tag{3.2}$$

where the integral is the Hukuhara integral. Observe that $X(t)$ is a solution of (3.1) if only if it satisfies (3.2) on I.

We recall some results in [1-8].

The local existence result on solution of SDE is the following.

Theorem 3.1. *Assume that*

(i) $F \in C\left[R_0, K_c(R^n)\right]$, $D\left[F(t,X(t)),\theta^n\right] \le M_0$, *on* $R_0 = I \times B(X_0,b)$ *where*

$$B(X_0,b) = \left\{X(t) \in K_c(R^n) : D\left[X(t),X_0\right] \le b\right\} \text{ and}$$

(ii) $g \in C\left[I \times [0,2b], R_+\right]$, $0 \le g(t,w) \le M_1$ *on* $I \times [0,2b], g(t,0) = 0$, $g(t,w)$ *is nondecreasing in w for each* $t \in I$ *and* $w(t) \equiv 0$ *is the unique solution of*

$$w' = g(t,w),\ w(t_0) = 0 \text{ on } I. \tag{3.3}$$

(iii) $D\left[F(t,\overline{X}(t)),F(t,X(t))\right] \le g\left(t, D\left[\overline{X}(t),X(t)\right]\right)$ *on* R_0.

Then, the (3.1) has a unique solution $X(t) = X(t,X_0)$ *on* $[t_0, t_0+\eta]$, *where* $\eta = min\left\{a, \frac{b}{M}\right\}$, $M = max\left\{M_0, M_1\right\}$.

The global existence result on solution of SDE is as follows.

Theorem 3.2. *Assume that* $F \in C\left[R_+ \times K_c(R^n), K_c(R^n)\right]$ *and*

$$D\left[F(t,X(t)),\theta^n\right] \le g(t, D\left[X(t),\theta^n\right]),\ (t,X(t)) \in R_+ \times K_c(R^n),$$

where $g \in C\left[R_+^2, R_+\right]$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in R_+$ *and the maximal solution* $r(t,t_0,w_0)$ *of*

$w' = g(t,w)$, $w(t_0) = w_0 \ge 0$ *exists on* $[t_0,+\infty)$. *Suppose further that* F *is smooth enough to guarantee local existence of solution of (3.1) for any* $(t_0,X_0) \in R_+ \times K_c(R^n)$. *Then the largest interval of existence of any solution* $X(t) = X(t,t_0,X_0)$ *of (3.1) such that* $D\left[X_0,\theta^n\right] \le w_0$ *is* $[t_0,+\infty)$.

The below theorem is an existence result on solutions of SDE.

Theorem 3.3. *Assume that* $F \in C\left[R_+ \times K_c(R^n), K_c(R^n)\right]$ *and*

(i) $D\left[F(t,X(t)),\theta^n\right] \le g(t, D\left[X(t),\theta^n\right])$, $(t,X(t)) \in [t_0,T] \times K_c(R^n)$,

where $g \in C\left[[t_0,T] \times R_+, R_+\right]$, $g(t,w)$ *is nondecreasing in* (t,w);

(ii) *the maximal solution* $r(t,t_0,w_0)$ *of*

$$w' = g(t,w),\ w(t_0) = w_0 \ge 0$$

exists on I.

Then there exists a solution $X(t) = X(t,t_0,X_0)$ *to the (3.1) which satisfies*

$$D\left[X(t), X_0\right] \leq r(t, w_0), t \in I, \text{ where } w_0 = D\left[X_0, \theta^n\right].$$

Some results on comparison of solutions of SDE and stability of solutions of SDE were studied in [1, 4, 5].

4. MAIN RESULTS

In this paper, we provide a set control differential equation (SCDE) as following

$$D_H X(t) = F(t, X(t), U(t)),\ X(t_0) = X_0 \in K_c(R^n), \tag{4.1}$$

where $F : I \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$, state $X(t) \in K_c(R^n)$, control $U(t) \in K_c(R^p)$.

The $U : I \to K_c(R^p)$ is integrable, is called an admissible control. Let U be a set of all admissible controls. The mapping $X \in C^1\left[I, K_c(R^n)\right]$ is said to be a solution of (4.1) on *I* if it satisfies (4.1) on *I*. Since $X(t)$ is continuously differentiable, we have

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s) ds, t \in I.$$

We associate with the initial value problem (4.1) the following

$$X(t) = X_0 + \int_{t_0}^{t} F(s, X(s), U(s)) ds, t \in I$$

where the integral is the Hukuhara integral. Observe that $X(t)$ is a solution of (4.1) if only if it satisfies (4.2) on *I*.

Now, based on the theorems 3.1-3.3 of SDE we have some existence results on solutions of SCDE.

Firstly, we have a unique existence of solution of SCDE as following.

Theorem 4.1. *Assume that*

(i) $F \in C\left[R_0, K_c(R^n)\right]$, $D\left[F(t, X(t), U(t)), \theta^n\right] \leq M_0$, *on* $R_0 = I \times B(X_0, b) \times \mathrm{U}$,

where $B(X_0, b) = \left\{X(t) \in K_c(R^n) : D\left[X(t), X_0\right] \leq b\right\}$ *and*

(ii) $g \in C\left[I \times [0, 2b], R_+\right]$, $0 \leq g(t, w) \leq M_1$ *on* $I \times [0, 2b], g(t, 0) = 0$, $g(t, w)$ *is nondecreasing in w for each* $t \in I$ *and* $w(t) \equiv 0$ *is a unique solution of*

$$w' = g(t, w),\ w(t_0) = 0 \text{ on } I. \tag{4.3}$$

(iii) $D\left[F(t, \overline{X}(t), \overline{U}(t)), F(t, X(t), U(t))\right] \leq g\left(t, D\left[\overline{X}(t), X(t)\right]\right)$ *on* R_0.

Then, the (4.1) has a unique solution $X(t) = X(t, X_0, U(t))$ *on* $[t_0, t_0 + \eta]$, *where* $\eta = min\left\{a, \frac{b}{M}\right\}$, $M = max\{M_0, M_1\}$.

Proof. Function $U(t)$ is of variable t. Set $h(t, X(t)) = F(t, X(t), U(t))$ plays the role of function $F(t, X(t))$ in theorems 3.1 and consider $U(t)$ as parameter, then using theorems 3.1, we have theorem 4.1. □

Then, we have the global existence of solution of SCDE as below.

Theorem 4.2. *Assume that* $F \in C\left[R_+ \times K_c(R^n) \times K_c(R^p), K_c(R^n)\right]$ *and*

$$D\left[F(t, X(t), U(t)), \theta^n\right] \le g(t, D\left[X(t), \theta^n\right]),\ (t, X(t), U(t)) \in R_+ \times K_c(R^n) \times \mathrm{U},$$

where $g(t, w)$ *is nondecreasing in* w *for each* $t \in R_+$ *and the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t, w),\ w(t_0) = w_0 \ge 0$$

exists on $[t_0, +\infty)$. *Suppose further that f is smooth enough to guarantee local existence of solution of (4.1) for any* $(t_0, X_0, U(t_0)) \in R_+ \times K_c(R^n) \times \mathrm{U}$. *Then the largest interval of existence of any solution* $X(t) = X(t, t_0, X_0, U(t))$ *of (4.1) such that* $D\left[X_0, \theta^n\right] \le w_0$ *is* $[t_0, +\infty)$.

This theorem also holds for $J = \left[t_0, T\right], T > t_0$.

Proof. Using theorem 3.2 and the proof is similar the proof of theorem 4.1. □

We adapt theorem 3.3 of SDE to below theorem of SCDE.

Theorem 4.3. *Assume that* $F \in C\left[R_+ \times K_c(R^n) \times K_c(R^p), K_c(R^n)\right]$ *and*

(i) $D\left[F(t, X(t), U(t)), \theta^n\right] \le g(t, D\left[X(t), \theta^n\right])$,

$(t, X(t), U(t)) \in [t_0, T] \times K_c(R^n) \times \mathrm{U}$,

where $g \in C\left[[t_0, T] \times R_+, R_+\right]$, $g(t, w)$ *is nondecreasing in* (t, w);

(ii) *the maximal solution* $r(t, t_0, w_0)$ *of*

$$w' = g(t, w),\ w(t_0) = w_0 \ge 0$$

exists on I.

Then there exists a solution $X(t) = X(t, t_0, X_0, U(t))$ *to the (4.1) which satisfies*

$$D\left[X(t), X_0\right] \le r(t, w_0),\ t \in I,\ where\ w_0 = D\left[X_0, \theta^n\right].$$

For comparison solutions of SCDE we need the following assumption.

Assumption

$F : R \times K_c(R^n) \times K_c(R^p) \to K_c(R^n)$ satisfies the condition

$$D\left[F(t,\overline{X}(t),\overline{U}(t)),F(t,X(t),U(t))\right] \le c(t)\left\{ D\left[\overline{X}(t),X(t)\right] + D\left[\overline{U}(t),U(t)\right]\right\} \quad (4.4)$$

For $t \in I; \overline{X}(t), X(t) \in K_c(R^n); \overline{U}(t), U(t) \in K_c(R^p)$, where $c(t)$ is a positive and integralble on I.

Let $C = \int_{t_0}^{T} c(t)dt$. Because $c(t)$ is integrable on I, it is bounded almost everywhere by a positive constant K.

Theorem 4.4 *Suppose that F satisfies assumption (4.4) and $\overline{X}(t), X(t)$ are solutions of SCDE (3.1) starting at $\overline{X}_0, X_0$ and of $\overline{U}(t), U(t)$, respectively. Then one has*

$$D\left[\overline{X}(t),X(t)\right] \le \varepsilon \quad \text{if} \quad D\left[\overline{U}(t),U(t)\right] \le \delta(\varepsilon) \quad \text{and} \quad D\left[\overline{X}_0,X_0\right] \le \delta(\varepsilon) .$$

Proof. The solutions of SCDE (3.1) for controls $\overline{U}(t), U(t)$ originating at $\overline{X}_0, X_0$, respectively, are equivalent to the following integral forms

$$\overline{X}(t) = \overline{X}_0 + \int_{t_0}^{t} F(s,\overline{X}(s),\overline{U}(s))ds$$

$$X(t) = X_0 + \int_{t_0}^{t} F(s,X(s),U(s))ds .$$

We estimate

$$\begin{aligned}
&D\left[\overline{X}(t),X(t)\right] \\
&= D\left[\overline{X}_0 + \int_{t_0}^{t} F(s,\overline{X}(s),\overline{U}(s))ds, X_0 + \int_{t_0}^{t} F(s,X(s),U(s))ds\right] \\
&\le D\left[\overline{X}_0,X_0\right] + D\left[\int_{t_0}^{t} F(s,\overline{X}(s),\overline{U}(s))ds, \int_{t_0}^{t} F(s,X(s),U(s))ds\right] \\
&\le D\left[\overline{X}_0,X_0\right] + \int_{t_0}^{t} D\left[F(s,\overline{X}(s),\overline{U}(s)),F(s,X(s),U(s))\right]ds \\
&\le D\left[\overline{X}_0,X_0\right] + \int_{t_0}^{t} c(s)\left\{D\left[\overline{X}(s),X(s)\right] + D\left[\overline{U}(s),U(s)\right]\right\}ds
\end{aligned}$$

$$\leq D\left[\bar{X}_0, X_0\right] + \int_{t_0}^{t} c(s) D\left[\bar{X}(s), X(s)\right] ds + \int_{t_0}^{t} c(s) D\left[\bar{U}(s), U(s)\right] ds.$$

If $D\left[\bar{U}(t), U(t)\right] \leq \delta(\varepsilon)$ and $D\left[\bar{X}_0, X_0\right] \leq \delta(\varepsilon)$, then

$$D\left[\bar{X}(t), X(t)\right] \leq (K+1)\delta(\varepsilon) + \int_{t_0}^{t} c(s) D\left[\bar{X}(s), X(s)\right] ds.$$

Here we have used (2.5), (2.7).

Using Gronwall inequality, we have

$$D\left[\bar{X}(t), X(t)\right] \leq (K+1)\delta(\varepsilon) exp(C).$$

It follows the proof if we choose $0 < \delta(\varepsilon) \leq \dfrac{\varepsilon}{(K+1) exp(C)}$.

5. CONCLUSION

In this paper we gave a new concept of set control differential equation and studied its existence of solutions. A comparison of two solutions was considered. Some more results on existence and comparison of solutions of set control differential equation will be presented in next works.

SỰ TỒN TẠI NGHIỆM CỦA PHƯƠNG TRÌNH VI PHÂN ĐIỀU KHIỂN TẬP

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TÓM TẮT: *Gần đây, lĩnh vực phương trình vi phân đã được nghiên cứu một cách trừu tượng hơn. Thay vì khảo sát dáng điệu của một nghiệm, người ta đã khảo sát một bó nghiệm (tập hợp các nghiệm). Thay vì nghiên cứu một phương trình vi phân, người ta nghiên cứu một bao vi phân (xem [10]). Đặc biệt, người ta đã nghiên cứu phương trình vi phân đa trị (tập) là phương trình vi phân mà cả biến và đạo hàm của nó đều là các tập hợp (xem [1-8]). Trong bài báo này, chúng tôi tổng quát hoá phương trình vi phân đa trị thành phương trình vi phân điều khiển đa trị, trình bày sự tồn tại nghiệm và so sánh các nghiệm của nó. Bài báo này là sự tiếp nối của các công trình của chúng tôi về hướng nghiên cứu này (xem [11-15]).*

Từ khoá: *Phương trình vi phân, Phương trình vi phân tập; Phương trình vi phân điều khiển tập.*

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