A LOVAZS TYPE THEOREM

Trần Ngọc Danh

Department of Mathematics and Computing Science

College of Natural Sciences

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Abstract:

Let $C(n)$ be the set of all n-dimensional boolean vectors and $C(n,k)$ be the set of all $\mathbf{a} = (a_1, ..., a_n) \in C(n)$ such that $a_1 + ... + a_n = k$. For $\mathbf{a} \in C(n,k)$ let $\delta_i\mathbf{a}$ denotes the vector of $C(n-1)$ obtained from $\mathbf{a}$ by deleting the i-component of $\mathbf{a}$ The *shadow* of $\mathbf{a}$ is defined to be $\Delta\mathbf{a} = \{\delta_i\mathbf{a} : 1 \le i \le n \text{ and } a_i = 0\}$ and that of $A \in C(n,k)$ is $\Delta A = \cup_{\mathbf{a}\in A}\Delta\mathbf{a}$.

In this paper we will prove a Lovazs type theorem: If $A \in C(n,k)$ with $|A| = \binom{x}{k}$ then $|\Delta A| = \binom{x-1}{k}$, after showing that $|\Delta C(A)| \le |\Delta A|$ where $C(A)$ is the first $|A|$ vectors of $C(n,k)$ in $\vee$-order.

MỘT ĐỊNH LÝ KIỂU LOVAZS

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Tóm tắt:

Gọi $C(n)$ là tập hợp các vector Bool n-chiều và $C(n,k)$ là tập hợp tất cả các vector $\mathbf{a} = (a_1, ..., a_n) \in C(n)$ sao cho $a_1 + ... + a_n = k$. Với $\mathbf{a} \in C(n,k)$ gọi $\delta_i\mathbf{a}$ là vector cuả $C(n-1)$ có được từ $\mathbf{a}$ bằng cách bỏ đi thành phần thứ i cuả $\mathbf{a}$ *Bóng* cua $\mathbf{a}$ được định nghĩa là $\Delta\mathbf{a} = \{\delta_i\mathbf{a} : 1 \le i \le n \text{ và } a_i = 0\}$ và *bóng* cua $A \in C(n,k)$ là $\Delta A = \cup_{\mathbf{a}\in A}\Delta\mathbf{a}$.

Trong bài này, chúng tôi sẽ chứng minh một định lý kiểu Lovazs: nếu $A \in C(n,k)$ với $|A| = \binom{x}{k}$ thì $|\Delta A| = \binom{x-1}{k}$, sau khi đã chứng minh rằng $|\Delta C(A)| \le |\Delta A|$ với $C(A)$ là tập hợp $|A|$ vector đầu tiên cuả $C(n,k)$ trong $\vee$-thứ tự.

1 Introduction

Let $C(n)$ be the set of all n-dimensional boolean vectors and $C(n,k)$ be the set of all $\mathbf{a} = (a_1, ..., a_n) \in C(n)$ for which $a_1 + \ldots + a_n = k$. Denote by $\delta_i\mathbf{a}$ the vector $C(n-1)$ obtained from $\mathbf{a}$ by deleting the i-th component of $\mathbf{a}$. The *shadow* $\Delta\mathbf{a}$ of $\mathbf{a}$ is defined to be $\Delta\mathbf{a} = \{\delta_i\mathbf{a} : 1 \le i \le n \text{ and } a_i = 0\}$ and that of $A \subseteq C(n,k)$ is $\Delta A = \bigcup_{\mathbf{a}\in A} \Delta\mathbf{a}$.

Definition 1 ($\vee$**-order**) *We order $C(n,k)$ by putting $\mathbf{a} < \mathbf{b}$ if $a_i = 1 > b_i = 0$ where $i = \min\{j : a_j \ne b_j\}$.*

For $A \subseteq \mathcal{C}(n,k)$ let $C(A)$ denote the first $|A|$ vectors of $\mathcal{C}(n,k)$ in $\vee$-order. If $C(A) = A$ then we say that A is *compressed* or A is an *initial section* IS. Our results in this paper are the following

Theorem 1 *For $A \subseteq \mathcal{C}(n,k)$ we have $|\Delta C(A)| \le |\Delta A|$.*

Theorem 2 *Let $A \subseteq \mathcal{C}(n,k)$ with $|A| = \binom{x}{k}$ where x is real and $x \ge k$.*

Then $|\Delta A| \ge \binom{x-1}{k}$.

Theorem 1 is quite similar to what we have done in [2]. Theorem 2 is a Lovazs type theorem dealing with a collection of k-subsets of an n-set [1].

2 Proof of the theorems

For $\mathbf{a} \in \mathcal{C}(n,k)$ let $\vee\mathbf{a} = \max\{j : a_j = 0\}$ and $\mathbf{a}^* = \delta_h \mathbf{a}$ with $h = \vee\mathbf{a}$. By a simple calculation we get

Lemma 1 *For $\mathbf{a} \in \mathcal{C}(n,k)$ we have $\mathbf{a}^* = \max\{\Delta\mathbf{a}\}$. Further if $\mathbf{a} < \mathbf{b}$ then $\mathbf{a}^* \le \mathbf{b}^*$ $\forall \mathbf{a}, \mathbf{b} \in \mathcal{C}(n,k)$.*

Here and elsewhere, $A + B$ denotes $A \cup B$ if $A \cap B = \emptyset$, and if A is a subset of vectors and $t \in \{0,1\}$ then $At = \{(\mathbf{x}, t);\ \mathbf{x} \in A\}$

For $A \subseteq \mathcal{C}(n,k)$ let $A_{n:t} = \{(a_1, ..., a_n) \in A : a_n = t\}$ ($t = 0$ or $t = 1$) and we can write $A = M0 + N1$ where $M \subseteq \mathcal{C}(n-1,k)$ and $N \subseteq \mathcal{C}(n-1,k-1)$.

The following lemma is immediate

Lemma 2 *If A is an IS then so is ΔA. Further,*

$$\Delta A = \{\mathbf{a}^* : \mathbf{a} = (a_1, ..., a_n) \in A \text{ and } a_n = 0\} = \delta_n(A_{n:0}).$$

Thus if A is an IS and $A = G0 + H1$ then $\Delta A = G$.

Definition 2 *Let A, $B \subseteq \mathcal{C}(n,k)$ with $A = M0 + N1$ and $B = G0 + H1$. If $G = C(M)$ and $H = C(N)$ then B is said to be the part compression of A and written as $B = PC(A)$. If $PC(A) = A$ then A is called part compressed.*

Lemma 3 *Let $A = M0 + N1 \subseteq \mathcal{C}(n,k)$ and $B = PC(A) = G0 + H1$. If Theorem 1.1 is true in n-1 dimensions then $|\Delta B| \le |\Delta A|$.*

Proof. Put $\mathbf{m}_0 = \max\{B_{n:0}\}$ and $\mathbf{m}_1 = \max\{B_{n:1}\}$. We consider two cases.

Case 1. $\mathbf{m}_1 < \mathbf{m}_0$. It is easy to check that $\Delta B = G$, so $|\Delta B| = |G| = |M| \le |\Delta A|$.

Case 2. $\mathbf{m}_1 > \mathbf{m}_0$. After a short argument we get $G_{n-1:1} \subseteq (\Delta H)1$ and hence $\Delta B = (\Delta G)0 + (\Delta H)1$.

Since Theorem 1 is true in $n-1$ dimensions we have $|\Delta G| \leq |\Delta M|$ and $|\Delta H| \leq |\Delta N|$ so $|\Delta B| = |(\Delta G)0| + |(\Delta H)1| \leq |(\Delta M)0| + |(\Delta N)1| \leq |\Delta A|$ as required.

Now let $\mathbf{z}_p$ denote the zero vector of dimension $p \geq 1$. If $\mathbf{a} \in \mathcal{C}(n,k)$ with $a_n = 1$ and $a_{n-1} = 0$ then we can write $\mathbf{a} = (\mathbf{u}, \mathbf{z}_p, 1)$ for some p with $\mathbf{u} = \emptyset$ or $\mathbf{u}$ ends 1.

Definition 3 *Let $A \subseteq \mathcal{C}(n,k)$. Define*

$$\phi\mathbf{a} = \begin{cases} (\mathbf{u}, 1, \mathbf{z}_p) & \text{if } \mathbf{a} = (\mathbf{u}, \mathbf{z}_p, 1) \text{ and } (\mathbf{u}, \mathbf{z}_p, 1) \notin A, \\ \mathbf{a} & \text{if otherwise.} \end{cases}$$

Also $\phi A = \{\phi\mathbf{a} : \mathbf{a} \in A\}$ and we say that A is $\phi-$closed if $\phi A = A$.

Lemma 4 *If A is part compressed then $\Delta(\phi A) \subseteq \phi(\Delta A)$.*

Proof. As in [2].

Proofs of the theorems. Both theorems are true for $n = 2$ and for $k = 0, 1$, so we consider the induction steps.

Proof of Theorem 1. By Lemma 3 and Lemma 4 we may assume that A is PC and $\phi-$closed. We then change A to an initial section B with $|\Delta B| \leq |\Delta A|$ as we have done in [2], that is $|\Delta C(A)| \leq |\Delta A|$.

Proof of Theorem 2. By Theorem 1 we can assume that $A = G0 + H1$ is an IS. Let y and z be the reals such that $|G| = \binom{y}{k}$ and $|H| = \binom{z}{k-1}$, so

$$\binom{x}{k} = \binom{y}{k} + \binom{z}{k-1}. \tag{1}$$

By Lemma 2 we know that $\Delta A = G$, so $|\Delta A| = |G| = \binom{y}{k}$.

If $y \geq x-1$ then $\binom{x-1}{k} \geq \binom{y}{k}$, so $|\Delta A| \geq \binom{x-1}{k}$. Now suppose that $y < x-1$. The induction hypothesis and the inequality $(\Delta G)0 + (\Delta H)1 \subseteq \Delta A$ give

$$|\Delta A| \geq |\Delta G| + |\Delta H| \geq \binom{y-1}{k} + \binom{z-1}{k-1}. \tag{2}$$

Further $|\Delta A| = |G| = \binom{y}{k}$, and so $\binom{y}{k} \geq \binom{y-1}{k} + \binom{z-1}{k-1}$.

Using the usual identity $\binom{y}{k} = \binom{y-1}{k} + \binom{y-1}{k-1}$ with a short calculation we get $z \leq y < x-1$, and hence

$$\binom{x-1}{k-1} - \binom{y-1}{k-1} - \binom{z-1}{k-2} > \binom{y}{k-1} - \binom{y-1}{k-1} - \binom{y-1}{k-2} \geq 0. \tag{3}$$

From (1) and (2) we get

$$|\Delta A| \geq \binom{y-1}{k} + \binom{z-1}{k-1} = \binom{y}{k} - \binom{y-1}{k-1} + \binom{z}{k-1} - \binom{z-1}{k-2}$$

or $$|\Delta A| \geq \binom{x}{k} - \binom{y-1}{k-1} - \binom{z-1}{k-2} \geq \binom{x-1}{k} + \binom{x-1}{k-1} - \binom{y-1}{k-1} - \binom{z-1}{k-2}$$

so by (3) we get $$|\Delta A| > \binom{x-1}{k}.$$

Thus Theorem 2 is proved.

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References

[1] I. Anderson, *Combinatorics of finite sets*, Clarandon Press, Oxford (1987). Press (1986).

[2] T.N. Danh and D.E. Daykin *Set of 0, 1 vectors with minimal sets of subvectors*, Rosrock. Math. Kollog., 50 (to appear)

[3] D.E. Daykin, *Ordered ranked posets, representation of integers and inequalities from extremal poset problems, Graphs and Order*, Proc. Conf. Banff, Canada (1984). Ed. I.Rival, pages 395-412.