Science & Technology Development Journal 2023, 26(3):2987-2995
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Natural Sciences: Research Article

Stable strong duality for a class of cone-constrained set-valued optimization problems: A perturbation approach

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ABSTRACT

In this paper we consider a general set-valued optimization problem of the model $\mathrm{WInf}_{x\in X} F(x)$*, where* $F\colon X \rightrightarrows Y \cup \{+\infty_Y\}$ *is a proper mapping. The problem is then embedded into a parametric problem and can be expressed as*

$$\text{(P)} \qquad \mathrm{WInf} \bigcup_{x\in X} \Phi(x, 0_Z)$$

where $\Phi\colon X \times Z \rightrightarrows Y^\bullet$ *is a proper set-valued perturbation mapping such that* $\Phi(x, 0_Z) = F(x)$*. A representation of the epigraph of the conjugate mapping* Φ^* *is established (Theorem 1) and it is used as the basic tool for establishing a general stable strong duality for the problem (P) (Theorem 3). As applications, the mentioned general strong duality result is then applied to a cone-constrained set-valued optimization problem (CSP) to derive three dual problems for (CSP): The Lagrange dual problem and two forms of Fenchel-Lagrange dual problems for (CSP). Consequently, three stable strong duality results for the three primal-dual pairs of problems are derived (Theorems 4, 5), among them, one is entirely new while the others extend some known ones in the literature.*

Keywords: set-valued optimization problems, perturbation mapping, perturbation approach to set-valued optimization problems, stable strong duality for set-valued optimization problems.

Introduction, Preliminaries, Notations and Basic Tools

Let X, Y, Z be lcHtvs with their topological dual spaces denoted by X^*, Y^* and Z^*, respectively. These dual spaces are endowed with their own corresponding weak*-topology. For a set $U \subset X$, we denote by $\operatorname{int} U$ and $\operatorname{lin} U$ the *interior* and the *linear hull* of U, respectively. We consider a general *set-valued optimimization problem*

$$\text{(P)} \quad \mathrm{WInf} \bigcup_{x\in X} F(x)$$

where $F\colon X \rightrightarrows Y \cup \{+\infty_Y\}$ is a proper mapping. The set-valued problem (P) is very general which includes many models of practical problem (see [2], [11] and references therein).

Like multi-optimization problems, different concepts of optimality can be used for (P). In this paper we focus on weakly efficient solutions because they are preferable from computational aspects and also because of the fact that they allow to apply the elegant conjugate duality theory which is the main tool of this paper.

Conjugate duality become a popular method to study scalar, vector optimization problems, for instance, in [1]-[9], [10], [11], [13], [14] and for set-valued optimization problems in many works of the author R.I. Bot and his co-authors and are collected in the book [2].

The perturbation approach has been used for scalar optimization problems, vector optimization problems (see [1], [2], [7], [8] and references therein), and also for set-valued optimization problems such as in [2], [15], [16]. In this paper the conjugate duality method is used together with the perturbation approach to study the duality for set-valued optimization problems. We propose a new dual problem for (P) and prove several duality results for (P). Some among them are new and some extend some known duality results in [2].

Following the perturbation approach, we now embed (P) into a parametric problem

$$(\mathrm{P}_z) \qquad \mathrm{WInf} \bigcup_{x\in X} \Phi(x, z)$$

where z runs on the parameter space Z with $\Phi\colon X \times Z \rightrightarrows Y^\bullet$ being a proper set-valued perturbation mapping such that $\Phi(x, 0_Z) = F(x)$ (0_z denotes the null vector of Z). The dual problem of (P_{0_Z}) is [2, p.367]

$$\text{(D)} \qquad \mathrm{WSup} \bigcup_{T\in\mathcal{L}(X,Z)} [-\Phi^*(0_\mathcal{L}, T)].$$

The basic tool for our study in this paper is a representation of the epigraph of the conjugate mapping Φ^* established in Section 2. A general stable strong duality for the pair (P) − (D) is established in Section 3. In Section 4, the result is then applied to the class of cone-constrained set-valued optimization problems to derive three forms of dual problems: The Lagrange dual problem and two forms of Fenchel-Lagrange dual problems. Consequently, three stable strong duality results for the three primal-dual pairs of problems are derived.

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History

- Received: 2023-05-09
- Accepted: 2023-07-09
- Published: 2023-09-30

DOI
https://doi.org/10.32508/stdj.v26i3.4086

Check for updates

Cite this article: Nguyen D, Dang H L. **Stable strong duality for a class of cone-constrained set-valued optimization problems: A perturbation approach**. *Sci. Tech. Dev.*. 2023; 26(3):2987-2995.

Weak Ordering Generated by a Convex Cone

Let K be a proper closed and convex cone in Y with nonempty interior, i.e., $\operatorname{int} K \neq \emptyset$. It is worth observing that for such a cone it holds $K + \operatorname{int} K = \operatorname{int} K$, which is equivalent to

$$(y \in K \text{ and } y + y' \notin \operatorname{int} K) \implies y' \notin \operatorname{int} K.$$

We define a *weak ordering* in Y generated by K as follows: for all $y_1, y_2 \in Y$,

$$y_1 <_K y_2 \iff y_1 - y_2 \in -\operatorname{int} K.$$

In Y we sometimes also consider the usual *ordering* generated by the cone K, $\leqq_K$, which is defined by $y_1 \leqq_K y_2$ if and only if $y_1 - y_2 \in -K$, for $y_1, y_2 \in Y$.

The next lemma is useful in the sequel.

Lemma 1. [3, Lemma 2.1 (i)] *For all $y, y' \in Y$ and $k_0 \in \operatorname{int} K$, there is $\mu > 0$ such that $y - \mu k_0 <_K y'$.*

We enlarge Y by attaching a *greatest element* $+\infty_Y$ and a *smallest element* $-\infty_Y$ w.r.t. $<_K$, which do not belong to Y, and denote $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. We understand, by convention, that $-\infty_Y <_K y <_K +\infty_Y$ for each $y \in Y$ and

$$-(+\infty_Y) = -\infty_Y, \quad -(-\infty_Y) = +\infty_Y,$$
$$(+\infty_Y) + y = y + (+\infty_Y) = +\infty_Y, \ \forall y \in Y \cup \{+\infty_Y\},$$
$$(-\infty_Y) + y = y + (-\infty_Y) = -\infty_Y, \ \forall y \in Y \cup \{-\infty_Y\}.$$

Moreover, for $M \subset Y$,

$$M + \{-\infty_Y\} = \{-\infty_Y\} + M = \{-\infty_Y\},$$
$$M + \{+\infty_Y\} = \{+\infty_Y\} + M = \{+\infty_Y\}.$$

The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

The following notions ([2, Definition 7.4.1], [13]) will be used throughout the paper.

Definition 1. Let $M \subset Y^\bullet$.

a) An element $\bar{v} \in Y^\bullet$ is said to be a *weakly infimal element* of M if

$$\begin{cases} v \not<_K \bar{v}, \ \forall v \in M, \\ \forall \tilde{v} \in Y^\bullet, \ \bar{v} <_K \tilde{v}, \ \exists v \in M : v <_K \tilde{v}. \end{cases}$$

b) An element $\bar{v} \in Y^\bullet$ is said to be a *weakly supremal element* of M if

$$\begin{cases} \bar{v} \not<_K v, \ \forall v \in M, \\ \forall \tilde{v} \in Y^\bullet, \ \tilde{v} <_K \bar{v}, \ \exists v \in M : \tilde{v} <_K v. \end{cases}$$

c) The set of all weakly infimal elements of M (weakly supremal elements of M, resp.) is called the *weak infimum* of M (the *weak supremum* of M, resp.) denoted by $\operatorname{WInf} M$ ($\operatorname{WSup} M$, resp.).

d) The *weak minimum* of M is the set $\operatorname{WMin} M = M \cap \operatorname{WInf} M$ and its elements are called the *weakly minimal elements* of M. Similarly, the *weak maximum* of M is $\operatorname{WMax} M = M \cap \operatorname{WSup} M$ and its elements are the *weakly maximal elements* of M.

The next properties related to the sets weak infimum, weak minimum, weak supremum, and weak maximum of a subset M of $Y^\bullet$ are traced out from [2, 4, 6] and [7, Proposition 2.1].

Proposition 1. *Assume further that $M \subset Y$ and $\operatorname{WSup} M \neq \{+\infty_Y\}$. Then it holds:*

(i) $\operatorname{WSup} M - \operatorname{int} K = M - \operatorname{int} K$,

(ii) *The following decomposition*[1] *of Y holds*

$$Y = (M - \operatorname{int} K) \cup \operatorname{WSup} M \cup (\operatorname{WSup} M + \operatorname{int} K),$$

(iii) $\operatorname{WSup} M = \operatorname{cl}(M - \operatorname{int} K) \setminus (M - \operatorname{int} K)$.

Remark 1. It is clear that $\operatorname{WInf} M = -\operatorname{WSup}(-M)$ for all $M \subset Y^\bullet$ and so, Proposition 1 holds true also when WSup, $+\infty_Y$, K, and $\operatorname{int} K$ are replaced by WInf, $-\infty_Y$, $-K$, and $-\operatorname{int} K$, respectively.

The Structure $(\mathcal{P}_p(Y)^\bullet, \preccurlyeq_K)$

Let $\mathcal{P}_0(Y^\bullet)$ be the collection of all non-empty subsets of $Y^\bullet$. The ordering "$\preccurlyeq_K$" on $\mathcal{P}_0(Y^\bullet)$ is defined [6] as, for $M, N \in \mathcal{P}_0(Y^\bullet)$,

$$M \preccurlyeq_K N \iff (v \not<_K u, \ \forall u \in M, \ \forall v \in N). \tag{1}$$

Other orderings on $\mathcal{P}_0(Y^\bullet)$ are also proposed in the literature (see, e.g., [11]).

Proposition 2. [6, Proposition 2] *For all $M, N \in \mathcal{P}_0(Y^\bullet)$, if $M \preccurlyeq_K N$ then $\operatorname{WSup} M \preccurlyeq_K \operatorname{WInf} N$.*

Definition 2. We say that a subset $U \subset Y$ is

[1] By the term "decomposition", we mean that the subsets in the right hand side are disjoint.

a (Y, K)-*partition style set* if the following decomposition of Y holds

$$Y = (U - \operatorname{int} K) \cup U \cup (U + \operatorname{int} K).$$

The collection of all (Y, K)-partition style subsets of Y is denoted by $\mathcal{P}_p(Y)$.

Denote also, $\mathcal{P}_p(Y)^\bullet := \mathcal{P}_p(Y) \cup \{\{+\infty_Y\}, \{-\infty_Y\}\}$. Clearly that if $M \subset Y^\bullet$ then $\pm\,\mathrm{WSup}\, M \in \mathcal{P}_p(Y)^\bullet$, $\pm\mathrm{WInf}M \in \mathcal{P}_p(Y)^\bullet$ and (by (1)), for any $U \in \mathcal{P}_p(Y)$, one has $U \preccurlyeq_K \{+\infty_Y\}$ and $\{-\infty_Y\} \preccurlyeq_K U$. Moreover, $(\mathcal{P}_p(Y)^\bullet, \preccurlyeq_K)$ is a partially ordered space [6]. It is worth noting that in $\mathcal{P}_p(Y)^\bullet, \preccurlyeq_K$ coincides with the ordering relation $\preccurlyeq$ introduced in [15] and [16].

Proposition 3. [7, Proposition 3.2, Lemma 3.1] *Let $U, V \in \mathcal{P}_p(Y)$. Then*

(i) *If $U \subset V$ then $U = V$,*

(ii) *The following decompositions hold:*

$$Y = (U - \operatorname{int} K) \cup (U + K) = (U - K) \cup (U + \operatorname{int} K),$$

(iii) $\mathrm{WSup}\, U = \mathrm{WInf} U = U$.

Set-valued mappings and Representation of the Conjugates of Perturbation Mappings

Let $F\colon X \rightrightarrows Y^\bullet$ be a set-valued mapping. For any nonempty subset $D \subset X$, we write $F(D) = \bigcup_{x\in D} F(x)$. The *domain* and the K-epigraph of F are defined by, respectively

$$\begin{aligned} \operatorname{dom} F &:= \big\{x \in X : F(x) \neq \emptyset \text{ and } F(x) \not\ni +\infty_Y\big\}, \\ \operatorname{epi}_K F &:= \{(x, y) \in X \times Y : y \in F(x) + K\}. \end{aligned}$$

For (a single-valued mapping) $f\colon X \to Y^\bullet$, the sum $f + F$ stands for a set-valued mapping from X into $Y^\bullet$ defined by $(f + F)(x) := \{f(x) + y : y \in F(x)\}$, for each $x \in X$. The next notations are quoted from [2].

Definition 3. Let C be a non-empty subset of X.

a) We say that F is *proper* if $-\infty_Y \notin F(X)$ and $\operatorname{dom} F \neq \emptyset$.

b) We say that F is K-*convex* if $\operatorname{epi}_K f$ is a convex subset of $X \times Y$.

c) We say that F is K-*convexlike on* C if for all $y_1, y_2 \in F(C) \cap Y$ and $\mu \in (0, 1)$ there is $\bar{x} \in C$ such that $\mu y_1 + (1-\mu) y_2 \in F(\bar{x}) + K$.

d) We say that F is *weakly K-upper bounded on* C if there is $\bar{y} \in Y$ such that $\bar{y} \in F(x) + K$ for all $x \in C$.

e)We say that F is *lower continuous at* $x_0 \in X$ if for all open set $D \subset Y$ satisfying $D \cap F(x_0) \neq \emptyset$ there exists a neighborhood U of x_0 such that $F(x) \cap D \neq \emptyset$ for all $x \in U$.

Associated with F, we define another set-valued map $F^*: \mathcal{L}(X, Y) \rightrightarrows Y^\bullet$ defined by

$$F^*(L) := \mathrm{WSup}[(L - F)(X)]$$

which is called the *conjugate* (also, conjugate mapping) of F (see [2, p.363], [13], [14]).

Proposition 4. *Assume that $F\colon X \rightrightarrows Y^\bullet$ is proper and $(L, y) \in \mathcal{L}(X, Y) \times Y$. Then*

$$(L, y) \in \operatorname{epi}_K F^* \iff \Big(y - L(x) \notin -F(x) - \operatorname{int} K,\ \forall x \in X\Big).$$

Proof. Without loss of generality, we can assume that $F^*(L) \neq \{+\infty\}$. Then, as $F^*(L) = \mathrm{WSup}[(L - F)(X)] \in \mathcal{P}_p(Y)^\bullet$, the next decompositions hold

$$\begin{aligned} Y &= (F^*(L) + K) \cup \{\mathrm{WSup}[(L - F)(X)] - \operatorname{int} K\} \\ &= (F^*(L) + K) \cup \{[(L - F)(X)] - \operatorname{int} K\} \end{aligned} \tag{2}$$

(by Proposition 3 (iii) and Proposition 1(i)).

We now have

$$\begin{aligned} &(L, y) \in \operatorname{epi} F^* \\ \iff &\quad y \in F^*(L) + K \\ \iff &\quad y \notin [(L - F)(X)] - \operatorname{int} K \quad \text{(by (2))} \\ \iff &\quad y - L(x) \notin -F(x) - \operatorname{int} K, \quad \forall x \in X, \end{aligned}$$

which completes the proof. □

We now consider a proper set-valued perturbation mapping $\Phi\colon X \times Z \rightrightarrows Y^\bullet$ (the name of this mapping will be explained in the next section). Note that $\Phi^* : \mathcal{L}(X, Y) \times \mathcal{L}(X, Z) \rightrightarrows$

$Y^\bullet$ and in the sequel, the conjugate of the mapping $\Phi(\cdot, 0_Z)\colon X \rightrightarrows Y^\bullet$ will be denoted by $\Phi(\cdot, 0_Z)^*$ (instead of $(\Phi(\cdot, 0_Z))^*$).

The mapping $\Phi\colon X \times Z \rightrightarrows Y^\bullet$ is called K-*convexlike-convex* if for all $z_1, z_2 \in Z$, $y_1 \in \Phi(X, z_1)$, $y_2 \in \Phi(X, z_2)$, and $\mu \in (0,1)$ there is $\bar{x} \in X$ such that

$$\mu y_1 + (1-\mu) y_2 \in \Phi(\bar{x}, \mu z_1 + (1-\mu) z_2) + K.$$

We now introduce a *qualifying condition* associated with Φ which will be used throughout this paper.

(CQ) $\exists \hat{x} \in X$ such that $\Phi(\hat{x}, .)$ is weakly $K-$ upper bounded on some neighborhood of 0_Z.

The next theorem is the basic tool in establishing the main results of this paper.

Theorem 1. *Assume that Φ is K-convexlike-convex and (CQ) fulfills. Then, it holds*

$$\operatorname{epi}_K \Phi(., 0_Z)^* = \bigcup_{T \in \mathcal{L}(X,Z)} \operatorname{epi}_K \Phi^*(., T). \tag{3}$$

Proof of the inclusion `` $\supset$ ''. Take $(L, y) \in \bigcup_{T \in \mathcal{L}(X,Z)} \operatorname{epi}_K \Phi^*(\cdot, T)$. Then, there is $T \in \mathcal{L}(X, Z)$ such that $(L, y) \in \operatorname{epi}_K \Phi^*(\cdot, T)$, or equivalently, $(L, T, y) \in \operatorname{epi}_K \Phi^*$, and hence, by Proposition 4,

$$y - L(x) - T(z) \notin -\Phi(x, z) - \operatorname{int} K, \ \forall (x, z) \in X \times Z.$$

In particular, when $z = 0_Z$, one has $y - L(x) \notin -\Phi(x, 0_Z) - \operatorname{int} K$ for all $x \in X$, which, again by Proposition 4, yields $(L, y) \in \operatorname{epi}_K \Phi^*(\cdot, T)$.

Proof of the inclusion `` $\subset$ ''. Take $(\bar{L}, \bar{y}) \in \operatorname{epi}_K \Phi(., 0_Z)^*$. By Proposition 4,

$$\bar{y} \notin \bar{L}(x) - \Phi(x, 0_Z) - \operatorname{int} K, \quad \forall x \in X. \tag{4}$$

We will show that there exists $\bar{T} \in \mathcal{L}(X, Z)$ such that $(\bar{L}, \bar{y}) \in \operatorname{epi}_K \Phi^*(., \bar{T})$. Set

$$\Delta_{\bar{L}} := \bigcup_{(x,z) \in \operatorname{dom} \Phi} \left(\left(\bar{L}(x) - \Phi(x, z) - K \right) \times \{z\} \right).$$

As Φ is K-convexlike-convex, it is easy to check that $\Delta_{\bar{L}}$ is a convex subset of $Y \times Z$.

Step 1. Prove that $\operatorname{int} \Delta_{\bar{L}} \neq \emptyset$. As (CQ) holds, there exist an open neighborhood $\hat{V}$ of $\hat{z}$ and $\hat{y} \in Y$ such that

$$\begin{aligned}
& \hat{y} \in \Phi(\hat{x}, z) + K, \quad \forall z \in \hat{V} \\
\Longrightarrow\ & \bar{L}(\hat{x}) - \hat{y} \in \bar{L}(\hat{x}) - \Phi(\hat{x}, z) - K, \quad \forall z \in \hat{V} \\
\Longrightarrow\ & \bar{L}(\hat{x}) - \hat{y} - \operatorname{int} K \subset \bar{L}(\hat{x}) - \Phi(\hat{x}, z) - K, \ \forall z \in \hat{V} \\
\Longrightarrow\ & (\bar{L}(\hat{x}) - \hat{y} - \operatorname{int} K) \times \hat{V} \subset \Delta_{\bar{L}} \\
\Longrightarrow\ & (\bar{L}(\hat{x}) - \hat{y} - \operatorname{int} K) \times \hat{V} \subset \operatorname{int} \Delta_{\bar{L}}.
\end{aligned} \tag{5}$$

So $\operatorname{int} \Delta_{\bar{L}} \neq \emptyset$ (as $\hat{V} \ni 0_Z$ and $\operatorname{int} K \neq \emptyset$).

Step 2. Prove that $(\bar{y}, 0_Z) \notin \operatorname{int} \Delta_{\bar{L}}$. Assume on the contrary that $(\bar{y}, 0_Z) \in \operatorname{int} \Delta_{\bar{L}}$. Then, there exists a neighborhood $U \times V$ of $(0_Y, 0_Z)$ such that $(\bar{y} + U) \times V \subset \Delta_{\bar{L}}$. If we take $\bar{k} \in U \cap \operatorname{int} K$ then one gets $(\bar{y} + \bar{k}, 0_Z) \in \Delta_{\bar{L}}$. This yields the existence of $(\bar{x}, 0_Z) \in \operatorname{dom} \Phi$ such that $\bar{y} + \bar{k} \in \bar{L}(\bar{x}) - \Phi(\bar{x}, 0_Z) - K$, leading to $\bar{y} \in \bar{L}(\bar{x}) - \Phi(\bar{x}, 0_Z) - \operatorname{int} K$, which contradicts (4). Thus, $(\bar{y}, 0_Z) \notin \operatorname{int} \Delta_{\bar{L}}$.

Step 3. As $(\bar{y}, 0_Z) \notin \operatorname{int} \Delta_{\bar{L}}$, apply the convex separation theorem ([12, Theorem 3.4]) to the point (singleton) $(\bar{y}, 0_Z)$ and the convex set $\Delta_{\bar{L}}$ in the space $Y \times Z$, one gets $(y_0^*, z_0^*) \in Y^* \times Z^*$ such that

$$y_0^*(\bar{y}) < y_0^*(y) + z_0^*(z), \quad \forall (y, z) \in \operatorname{int} \Delta_{\bar{L}}, \tag{6}$$

and consequently,

$$y_0^*(\bar{y}) \leq y_0^*(y) + z_0^*(z), \quad \forall (y, z) \in \Delta_{\bar{L}}. \tag{7}$$

Next, we show that

$$y_0^*(k') < 0, \quad \forall k' \in \operatorname{int} K. \tag{8}$$

Indeed, take $k' \in \operatorname{int} K$. With $\bar{L}(\hat{x}) - \hat{y} \in Y$ (appear in Step 1) and $k', \bar{y} \in Y$, by Lemma 1, there is $\mu > 0$ such that $\bar{y} - \mu k' \in \bar{L}(\hat{x}) - \hat{y} - \operatorname{int} K$. Hence, by (5), $(\bar{y} - \mu k', 0_Z) \in \operatorname{int} \Delta_L$. In turn, (6) leads to $y_0^*(\bar{y}) < y_0^*(\bar{y} - \mu k') + z_0^*(0_Z)$, or $y_0^*(k') < 0$, and (8) holds.

Step 4. Take $k_0 \in \operatorname{int} K$ such that $y_0^*(k_0) = -1$ (it is possible by (8)) and $T\colon Z \to Y$ defined by $T(z) = -z_0^*(z)k_0$ for all $z \in Z$ (z_0^* exists by the separation theorem in Step 3). It is easy to see that $T \in \mathcal{L}(Z,Y)$ and for all $z \in Z$, it holds $\big(y_0^* \circ T\big)(z) = y_0^*(-z_0^*(z)k_0) = -y_0^*(k_0)z_0^*(z) = z_0^*(z)$. Thus, (7) can be rewritten as

$$y_0^*(\bar y) \le y_0^*(y) + (y_0^* \circ T)(z), \quad \forall (y,z) \in \Delta_{\bar L},$$

or equivalently,

$$y_0^*(y + T(z) - \bar y) \ge 0, \quad \forall (y,z) \in \Delta_{\bar L}.$$

So, by (8), $y + T(z) - \bar y \notin \operatorname{int} K$, yielding

$$\bar y \notin y + T(z) - \operatorname{int} K, \quad \forall (y,z) \in \Delta_{\bar L}. \tag{9}$$

Now, as $\big(\bar L(x) - \Phi(x,z), z\big) \in \Delta_{\bar L}$ for all $(x,z) \in \operatorname{dom}\Phi$, it follows from (9) that

$$\bar y \notin \bar L(x) - \Phi(x,z) + T(z) - \operatorname{int} K, \ \forall (x,z) \in \operatorname{dom}\Phi. \tag{10}$$

By Proposition 4, $(\bar L, T, \bar y) \in \operatorname{epi}_K \Phi^*$, or equivalently $(\bar L, \bar y) \in \operatorname{epi}_K \Phi^*(.,T)$ and we are done. □

Duality for Set-Valued Optimization Problems: A Perturbation Approach

Let X, Y, Z and K be as in Section 2. The same way as with Y, we also enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$, which do not belong to Z, and define $Z^\bullet := Z \cup \{-\infty_Z, +\infty_Z\}$.

We consider a general *set-valued otimization problem* of the model

$$(\mathrm{P}) \quad \operatorname{WInf} \bigcup_{x\in X} F(x)$$

where $F\colon X \rightrightarrows Y \cup \{+\infty_Y\}$ is a proper mapping. The set-valued problem (P) is very general which includes many practical problems (set-valued, vector-valued) problems (see [2], [11] and references therein).

We now embed (P) into a parametric problem

$$(\mathrm{P}_z) \qquad \operatorname{WInf} \bigcup_{x\in X} \Phi(x,z)$$

where z runs on the parameter space Z with $\Phi\colon X \times Z \rightrightarrows Y^\bullet$ being a proper set-valued perturbation mapping such that $\Phi(x, 0_Z) = F(x)$ (0_z denotes the null vector of Z), and so, (P_{0_z}) coincides with (P) and it can be expressed as

$$(\mathrm{P}) \qquad \operatorname{WInf} \bigcup_{x\in X} \Phi(x,0_Z).$$

Assume that $\operatorname{dom}\Phi(., 0_Z) \neq \emptyset$.

The *dual problem* of (P) is [2, p.367]

$$(\mathrm{D}) \qquad \operatorname{WSup} \bigcup_{T\in\mathcal{L}(X,Z)} [-\Phi^*(0_{\mathcal{L}}, T)].$$

The *values* of (P) and (D) are denoted by val(P) and val(D), which means

$$\operatorname{val}(\mathrm{P}) = \operatorname{WInf} \bigcup_{x\in X} \Phi(x,0_Z)$$

and

$$\operatorname{val}(\mathrm{D}) = \operatorname{WSup} \bigcup_{T\in\mathcal{L}(X,Z)} [-\Phi^*(0_{\mathcal{L}}, T)].$$

We say that "strong duality holds for the pair $(\mathrm{P}) - (\mathrm{D})$" if $\operatorname{val}(\mathrm{P}) = \operatorname{val}(\mathrm{D})$ and (D) attains at any points from its value.

Denote by (P^L) the problem (P) perturbed by a linear operator $L \in \mathcal{L}(X,Y)$. Then (P^L) and its corresponding dual problem (D^L), are

$$(\mathrm{P}^L) \qquad \operatorname{WInf} \bigcup_{x\in X} [\Phi(x,0_Z) - L(x)],$$

$$(\mathrm{D}^L) \qquad \operatorname{WSup} \bigcup_{T\in\mathcal{L}(X,Z)} [-\Phi^*(L,T)].$$

It is said that *stable strong duality holds for the pair* $(\mathrm{P}) - (\mathrm{D})$ if the strong duality holds for the pair $(\mathrm{P}^L) - (\mathrm{D}^L)$ for all $L \in \mathcal{L}(X,Y)$.

The next theorem covers [2, Proposition 7.4.16] and can be seen as a perturbation extension of [15, Theorem 3.25] (or [16, Theorem 2.6]) .

Theorem 2 (Weak duality)**.** *For any $L \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Z,Y)$, it holds:*

(i) $-\Phi^*(L,T) \preccurlyeq_K \operatorname{val}(\mathrm{P}^L)$,

(ii) $\operatorname{val}(\mathrm{D}^L) \preccurlyeq_K \operatorname{val}(\mathrm{P}^L)$.

Proof. Firstly, let us denote

$$M := \bigcup_{T\in\mathcal{L}(X,Z)} \big(-\Phi^*(L,T)\big), \qquad N := \bigcup_{x\in X} [\Phi(x,0_Z) - L(x)]. \tag{11}$$

Then, it is easy to see that $\operatorname{val}(\mathrm{D}^L) = \operatorname{WSup} M$ and $\operatorname{val}(\mathrm{P}^L) = \operatorname{WInf} N$.

(i) Take $\bar x \in X$ and $\bar y \in -\Phi^*(L,T)$. As $-\bar y \in \Phi^*(L,T) = \operatorname{WSup}\{L(x) + T(z) - \Phi(x,z) : (x,z) \in X \times Z\}$, one has $-\bar y \notin L(x) + T(z) - \Phi(x,z) - \operatorname{int} K$ for all $(x,z) \in X \times Z$ (by Proposition 1(ii)), and hence, with $x = \bar x$ and $z = 0_Z$, one gets $\bar y \notin \Phi(\bar x, 0_Z) - L(\bar x) +$

int K. So, $-\Phi^*(L,T) \preccurlyeq_K N$. Consequently, $-\Phi^*(L,T) = \text{WSup}[-\Phi^*(L,T)] \preccurlyeq_K \text{WInf}N = \text{val}(\text{P}^L)$ (by Proposition 3 (iii) and Proposition 2).

(ii) It follows from (i) that $M \preccurlyeq_K \text{WInf}N$, and hence, one gets $\text{val}(\text{D}^L) = \text{WSup}\, M \preccurlyeq_K \text{WInf}[\text{val}(\text{P}^L)] = \text{val}(\text{P}^L)$ (again by Proposition 3 (iii)). □

Theorem 3 (Stable strong duality)**.** *Assume that Φ is K-convexlike-convex and (CQ) fulfills. Then, the stable strong duality holds for the pair* (P) − (D).

Proof. Take arbitrarily $L \in \mathcal{L}(X,Y)$. Denote $\text{WMax}(\text{D}^L) = \text{WMax} \bigcup_{T\in\mathcal{L}(X,Z)} [-\Phi^*(0_{\mathcal{L}},T)]$. We will show that

$$\text{val}(\text{P}^L) = \text{val}(\text{D}^L) = \text{WMax}(\text{D}^L). \tag{12}$$

Firstly, as $\text{dom}\,\Phi(.,0_Z) \neq \emptyset$, one has $\text{val}(\text{P}^L) \neq \{+\infty_Y\}$. If $\text{val}(\text{P}^L) = \{-\infty_Y\}$ then, by Theorem 2, $\text{val}(\text{D}^L) = \{-\infty_Y\}$, and so, $-\Phi^*(L,T) = \{-\infty_Y\}$ for $T \in \mathcal{L}(X,Z)$. Consequently, $\text{WMax}(\text{D}^L) = \{-\infty_Y\} = \text{val}(\text{VP}^L)$, and (12) holds.

Assume now that $\text{val}(\text{P}^L) \subset Y$ and we will show that

$$\text{val}(\text{P}^L) \subset \text{WMax}(\text{D}^L). \tag{13}$$

Take $y \in \text{val}(\text{P}^L)$, we prove that $y \in \text{WMax}(\text{D}^L)$.

Let M and N be the sets as in (11) (in the proof of Theorem 2). It follows from Theorem 2 (i) that

$$M \preccurlyeq_K \text{WInf}N. \tag{14}$$

Next, since $y \in \text{val}(\text{P}^L) = \text{WInf}N$, by Proposition 1(ii) and Remark 1,

$$y \notin \Phi(x,0_Z) - L(x) + \text{int}\,K, \quad \forall x \in X,$$

which means $(L,-y) \in \text{epi}_K \Phi(.,0_Z)^*$ (see Proposition 4). Observing that, under the current assumption, (3) holds (see Theorem 1). So, $(L,-y) \in \bigcup_{T\in\mathcal{L}(X,Z)} \text{epi}_K \Phi^*(.,T)$, and hence, there is $\bar{T} \in L(Z,Y)$ such that

$$-y - L(x) - \bar{T}(z) \notin -\Phi(x,z) - \text{int}\,K, \forall (x,z) \in X \times Z,$$

which is equivalent to

$$\begin{aligned} -y &\notin \big\{L(x) + \bar{T}(z) - \Phi(x,z) : (x,z) \in X \times Z\big\} - \text{int}\,K \\ &= \Phi^*(L,\bar{T}) - \text{int}\,K. \end{aligned} \tag{15}$$

Now as $\Phi^*(L,\bar{T}) \in \mathcal{P}_p(Y)$,

$$Y = \big(\Phi^*(L,\bar{T}) - \text{int}\,K\big) \cup \big(\Phi^*(L,\bar{T}) + K\big),$$

which, together with (15), yields

$$-y \in \Phi^*(L,\bar{T}) + K. \tag{16}$$

Combine (14) and (16), one gets

$$-y \in (\Phi^*(L,\bar{T}) + K) \setminus (\Phi^*(L,\bar{T}) + \text{int}K) = \Phi^*(L,\bar{T})$$

(by Proposition 1(iii)), and hence, $y \in -\Phi^*(L,\bar{T}) \subset M$. This, together with (14), yields $y \in M \setminus (M - \text{int}\,K) = \text{WMax}\,M = \text{WMax}(\text{D}^L)$ (again by Proposition 1(iii)). So, (13) has been proved and then

$$\text{val}(\text{P}^L) \subset \text{WMax}(\text{D}^L) \subset \text{WSup}(\text{D}^L). \tag{17}$$

This leads to $\text{val}(\text{P}^L) = \text{val}(\text{D}^L)$ (by Proposition 3 (i) and the fact that $\text{val}(\text{P}^L), \text{val}(\text{D}^L) \in \mathcal{P}_p(Y)^\bullet$) and we obtain $\text{val}(\text{P}^L) = \text{WMax}(\text{D}^L)$. □

Remark 2. Theorem 3 is a stable extension of Theorems 7.4.20 and 7.4.27 in [2]. Observing that if (CQ) holds then the set-valued $\Psi\colon Z \rightrightarrows Y^\bullet$, with $\Psi(z) := \bigcup_{x\in X} \Phi(x,z)$ is K-upper bounded on some neighborhood of 0_Z and $0_Z \in \text{int}(\text{dom}\,\Psi)$. So the assumption of Theorem 3 is slightly stronger than the one in [2, Theorem 7.4.27] while the strong duality in the conclusion of Theorem 3 is stable strong duality under linear perturbations (stronger than the conclusion in [2, Theorem 7.4.27]), and hence, this theorem somehow extends [2, Theorem 7.4.20].

Duality for Cone-Constrained Set-Valued Optimization Problems

In this section, we will apply the strong duality results for the set-valued problem (P) given in a perturbation form in the previous section to a class of *cone-constrained set-valued otimimization problems* of the model

$$\text{(CSP)}\quad \text{WMin}\bigcup_{x} F(x) \quad \text{subject to } x \in C,\ G(x)\cap(-S)\neq\emptyset,$$

where X, Y, Z are as in previous sections, $F\colon X \rightrightarrows Y\cup\{+\infty_Y\}$ and $G\colon X \rightrightarrows Z\cup\{+\infty_Z\}$ are proper mappings, C is a nonempty convex subset of X and S is a nonempty closed convex cone in Z. Let us denote

$$A = \{x\in C : G(x)\cap(-S)\neq\emptyset\}$$

and assume throughout this section that $A\cap \operatorname{dom} F\neq\emptyset$.

For $T\in\mathcal{L}(Z,Y)$ and $G\colon X\rightrightarrows Z\cup\{+\infty_Z\}$, the composite function $T\circ G\colon X\rightrightarrows Y^\bullet$ is defined by $(T\circ G)(x) := \{T(z) : z\in G(x)\}$ if $G(x)\subset Z$ and $(T\circ G)(x) := \{T(z) : z\in G(x)\cap Z\}\cup\{+\infty_Y\}$ if $G(x)\ni +\infty_Z$.

The *cone of positive operators* and *the cone of weakly positive operators* from Z to Y [4] are defined respectively

$$\mathcal{L}_+(S,K) := \{T\in\mathcal{L}(Z,Y) : T(S)\subset K\},$$
$$\mathcal{L}_+^w(S,K) := \{T\in\mathcal{L}(Z,Y) : T(S)\cap(-\operatorname{int} K)=\emptyset\}.$$

When $Y=\mathbb{R}$, both these cones collapse to the usual (positive) dual cone S^+ of S.

Now, as illustrations, we will give three ways to define perturbation mappings for (CSP). The first two ones were already proposed in [2, Section 7.2.1]. As it is shown below, each perturbation mapping will define a dual problem and so, with these three perturbation mappings, when applying to the problem (CSP), the stable strong duality results in Section 3 (Theorem 3), one gets three corresponding stable strong duality results for (CSP).

• *The Lagrange perturbation mapping:* Take Z as the space of perturbation variables and $\Phi_1\colon X\times Z\rightrightarrows Y\cup\{+\infty_Y\}$ by

$$\Phi_1(x,z) = \begin{cases} F(x), & \text{if } x\in C \text{ and } z\in -G(x)-S,\\ +\infty_Y, & \text{otherwise.}\end{cases}$$

Associated with the perturbation mapping Φ_1 is the *Lagrange dual problem* of (CSP)

$$\text{(CSD}^1\text{)}\quad \underset{T\in\mathcal{L}_+^w(S,K)}{\text{WSup}}\ \underset{(x,s)\in C\times S}{\text{WInf}}\ \cup\,[F(x)+(T\circ G)(x)+T(s)].$$

• *The first Fenchel-Lagrange perturbation mapping:* Take $X\times Z$ as the space of perturbation variables and $\Phi_2\colon X\times Z\times Z\rightrightarrows Y\cup\{+\infty_Y\}$ by

$$\Phi_2(x,x',z) = \begin{cases} F(x+x'), & \text{if } x\in C,\, z\in -G(x)-S,\\ +\infty_Y, & \text{otherwise.}\end{cases}$$

This perturbation mapping leads to the *First Fenchel-Lagrange dual problem* of (CSP)

$$\text{(CSD}^2\text{)}\quad \underset{\substack{L'\in\mathcal{L}(X,Y)\\ T\in\mathcal{L}_+^w(S,K)}}{\text{WSup}}\ \text{WInf}\big[-F^*(L')-(I_C+T\circ G)^*(-L')-I_{-S}^*(T)\big]$$

• *The second Fenchel-Lagrange perturbation mapping:* Let $\widetilde{Z} := X\times X\times Z$ be the space of perturbation variables and we define the perturbation mapping $\Phi_3\colon X\times\widetilde{Z}\to Y\cup\{+\infty_Y\}$ as $\Phi_3(x,x',x'',z) = F(x+x')$ if $x+x''\in C$, $z\in -G(x)-S$ and $\Phi_3(x,x',x'',z) = +\infty_Y$, otherwise.

By an easy calculation, we get

$$\begin{aligned}\Phi_3^*(L,L',L'',T) = \quad & \text{WSup}\Big[F^*(L')+I_C^*(L'')+\\ & +(T\circ G)^*(L-L'-L'')\\ & +I_{-S}^*(T)\Big],\end{aligned}$$

for all $(L,L',L'',T)\in\mathcal{L}(X,Y)^3\times\mathcal{L}(X,Z)$.

Recall that if $T\notin\mathcal{L}_+^w(S,K)$ then $I_{-S}^*(T) = \{+\infty\}$ and hence $\Phi_3^*(L,L',L'',T) = \{+\infty\}$. So, the dual problem (D) with $\Phi=\Phi_3$ gives us a new dual problem of (CSP):

$$\text{(CSD}^3\text{)}\quad \underset{\substack{L',L''\in\mathcal{L}(X,Y)\\ T\in\mathcal{L}_+^w(S,K)}}{\text{WSup}}\ \text{WInf}\Big[-F^*(L')-I_C^*(L'') -(T\circ G)^*(-L'-L'')-I_{-S}^*(T)\Big].$$

This type of dual problem of (CSP), to the best of knowledge of the authors, has never appeared in any works for set-valued optimization problems in the literature, and so, the stable strong duality stated in Theorem 4 below is new. It is also worth observing that in the case where F and G are (single-value) vector-valued functions, (CSD^3) collapses to the dual problem (VD^3) in [7] for vector optimization problems.

Proposition 5. *It holds that* $\mathrm{val}(\mathrm{CSD}^3) \preccurlyeq_K \mathrm{val}(\mathrm{CSD}^2) \preccurlyeq_K \mathrm{val}(\mathrm{CSD}^1) \preccurlyeq_K \mathrm{val}(\mathrm{CSP})$.

Proof. Use the same argument as in the proof of [7, Proposition 6.1]. □

The qualifying condition (CQ) associated with the perturbation Φ, when specified to Φ_i, $i = 1, 2, 3$, turns to be the following ones:

(CQ_1) $\exists \hat{x} \in C \cap \mathrm{dom}\, F : G(\hat{x}) \cap \mathrm{int}(-S) \neq \emptyset$.

(CQ_2) $\exists \hat{x} \in C : G(\hat{x}) \cap \mathrm{int}(-S) \neq \emptyset$ and F is weakly K-upper bounded on some neighborhood of $\hat{x}$.

(CQ_3) $\exists \hat{x} \in \mathrm{int}\, C : G(\hat{x}) \cap \mathrm{int}(-S) \neq \emptyset$, F is weakly K-upper bounded on some neighborhood of $\hat{x}$ and G is lower continuous at $\hat{x}$.

It is easy to see that

$$(CQ_3) \Longrightarrow (CQ_2) \Longrightarrow (CQ_1). \qquad (18)$$

Theorem 4. *Let F be K-convex, G be S-convex, and assume that (CQ_3) holds. Then, stable strong duality holds for the pair* $(\mathrm{CSP}) - (\mathrm{CSD}^3)$.

Proof. It is worth observing firstly that when F is K-convex, G is S-convex and C is a convex subset of X then Φ_3 is K-convexlike-convex. We now assert some points:

• By the assumption (CQ_3), there are a convex neighborhood U_1 of 0_X and $\hat{y} \in Y$ such that

$$\hat{y} \in F(x) + K, \quad \forall x \in \hat{x} + U_1. \qquad (19)$$

• As $\hat{x} \in \mathrm{int}\, C$, there is a neighborhood U_2 of 0_X such that

$$\hat{x} + U_2 \subset C. \qquad (20)$$

• As $G(\hat{x}) \cap (-\,\mathrm{int}\, S) \neq \emptyset$, there exist $\hat{z} \in G(\hat{x})$ and an open neighborhood W of 0_Z such that

$$\hat{z} + W \subset -S. \qquad (21)$$

• On the other hand, as G is lower continuous at $\hat{x}$ and $\hat{z} + \frac{1}{2}W$ is an open subset of Z satisfying $(\hat{z} + \frac{1}{2}W) \cap G(\hat{x}) \neq \emptyset$, there is a neighborhood U_3 of 0_X such that $(\hat{z} + \frac{1}{2}W) \cap G(x) \neq \emptyset$ for all $x \in \hat{x} + U_3$. Then, for each $x \in \hat{x} + U_3$, there is $z_x \in G(x)$ such that $z_x \in \hat{z} + \frac{1}{2}W$, and hence

$$\begin{aligned} z_x + \frac{1}{2}W &\subset \hat{z} + \frac{1}{2}W + \frac{1}{2}W \\ &= \frac{1}{2}(\hat{z} + W) + \frac{1}{2}(\hat{z} + W) \subset -S \end{aligned}$$

(by (21) and the convexity of S). So

$$\frac{1}{2}W \subset -z_x - S \subset -G(x) - S, \quad \forall x \in \hat{x} + U_3. \qquad (22)$$

• Finally, take

$$V = [\hat{x} + (U_3 \cap \frac{1}{2}U_1)] \times (\frac{1}{2}U_1) \times U_2 \times \frac{1}{2}W.$$

For $(x, x', x'', z) \in V$, it follows from (20) and (22) that $\Phi_3(x, x', x'', z) = F(x + x')$. This, together with (20), yields $\hat{y} \in \Phi_3(x, x', x'', z) + K$.

We have just shown that Φ_3 is weakly K-upper bounded on some neighborhood of $(\hat{x}, 0_X, 0_X, 0_Z)$. This ensures that the qualifying (CQ) holds with $\Phi = \Phi_3$. The conclusion now follows from Theorem 3. □

Theorem 5. *The next assertions are true.*

(i) *If $F \times G$ is $K \times S$-convexlike on C and (CQ_1) holds then the stable strong duality holds for the pair* $(\mathrm{CSP}) - (\mathrm{CSD}^1)$.

(ii) *If F is K convex, G is S-convex, C is a convex set, and (CQ_2) holds then the stable strong duality holds for the pair* $(\mathrm{CSP}) - (\mathrm{CSD}^2)$.

Proof. Use a similar argument as in the proof of Theorem 4. □

Remark 3. (a) Theorem 5 (i) and (ii) can be considered as the stable versions of [2, Theorems 7.5.6 and 7.5.10], respectively. On the other hand, Theorem 5 (i) is similar to [15, Theorem 3.26] or [16, Theorem 2.7]. However, the dual problem defined in these works is a bit different from (CSD^1). There, the weak supremum was taken operators T of the special form $T(z) = \langle \lambda, z \rangle c$ with $\lambda \in Z^*$ and $c \in \mathrm{int}\, S$ instead of $T \in \mathcal{L}(X, Z)$ as in (CSD^1) (i.e, in these works, it is also required that $\mathrm{int}\, S \neq \emptyset$).

(b) As (18) always holds, if (CQ_3) holds, then (by Theorem 5), the stable strong duality holds for both the pairs $(\mathrm{CSP}) - (\mathrm{CSD}^1)$ and $(\mathrm{CSP}) - (\mathrm{CSD}^2)$.

ACKNOWLEDGMENT

None.

AUTHOR CONTRIBUTION

All authors made important contributions to the analysis and interpretation of the results in the manuscript. All authors read and approved the final manuscript.

COMPETING INTERESTS

The authors declare no competing financial interests.

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